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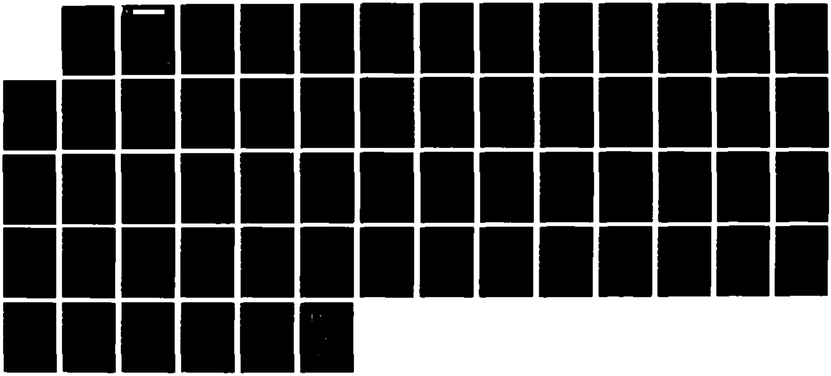

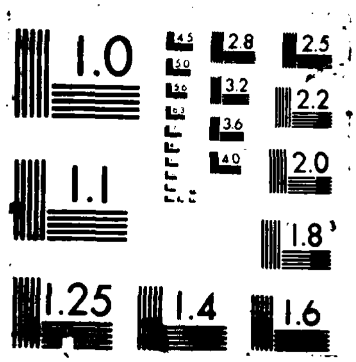
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THE OPTIMAL CONVERGENCE RATE OF THE p-VERSION
OF THE FINITE ELEMENT METHOD

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The p-Version of the finite element method has been previously analyzed for elliptic problems with homogeneous boundary conditions. For a homogeneous condition of the Dirichlet type, it was shown that the exponential asymptotic convergence rate was optimal up to an arbitrarily small positive parameter epsilon. In this paper, an alternate proof is discussed which yields a better estimate by removing the dependence on epsilon. The analysis is extended to treat problems with inhomogeneous boundary conditions of both the Dirichlet and Neumann type

Estimates for the case when the solution has singularities at the corners of the domain are also provided.

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THE OPTIMAL CONVERGENCE RATE OF THE p-VERSION
OF THE FINITE ELEMENT METHOD

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1. INTRODUCTION

There are three versions of the finite element method: the h-version, the p-version and the h-p version. The h-version is the standard one, where the degree p of the elements is fixed, usually on low level, typically $p = 1,2,3$ and the accuracy is achieved by properly refining the mesh. The p-version, in contrast, fixes the mesh and achieves the accuracy by increasing the degrees p of the elements uniformly or selectively. The h-p version is the combination of both.

The standard h-version has been thoroughly investigated, and many programs are available, both commercial and research codes. The p-version and h-p versions are new developments. There is only one commercial code, the system PROBE (Noetic Tech, St. Louis). The theoretical aspects have been studied only recently. The first theoretical paper appeared in 1981 (see [7]). See also [1], [7], [8], [9], [10], [11]. For the numerical, computational and implementational aspects we refer to [2].

In [4] it has been shown that the rate of convergence is an optimal one up to an arbitrarily small $\varepsilon > 0$, namely

$$(1.1) \qquad \|e\|_{H^1} \leq C(\varepsilon)p^{-(k-1)+\varepsilon}\|u\|_{H^k}$$

In the case when the solution has singular behaviour of the type $u = r^{\alpha}$, $\alpha > 0$, ((r,θ) being polar coordinates) and the vertex of the elements is at the origin, then

$$(1.2) \qquad \|e\|_{H^1} \leq C(\varepsilon)p^{-2\alpha+\varepsilon}.$$

The p-version has the rate of convergence which is twice the rate of the h-version with uniform mesh.

The proof of [4] indicates that the term $C(\varepsilon)$ can grow quickly with $\varepsilon \to 0$. Nevertheless, computational experience indicates that (1.1) and (1.2) hold without the term ε, i.e. it suggests

$$(1.3) \qquad \|e\|_{H^1} \le Cp^{-(k-1)}\|u\|_{H^k}$$

and

$$(1.4) \qquad \|e\|_{H^1} \le Cp^{-2\alpha},$$

respectively.

We show in this paper that in fact (1.3) and (1.4) hold and the term ε in (1.1) and (1.2) appeared only due to technicalities in the original proof.

In [4], only the case when essential boundary conditions are homogeneous was addressed. In this paper we deal with the general case. The ideas and techniques of this paper differ significantly from what was used in [4]. Section 2 addresses the preliminaries and basic notions. Section 3 deals with approximation properties of polynomials on a square. Section 4 analyzes the rate of convergence under the assumption that the solution does not have singular behaviour and proves (1.3) for homogeneous and nonhomogeneous essential boundary conditions. Section 5 deals with the case when the solution has a singular behaviour and proves (1.4). Section 6 summarizes the results and addresses briefly various generalizations.

2. PRELIMINARIES

2.1. Notation

By R^2 we denote the usual Euclidean space with $x = (x_1,x_2) \in R^2$. By $\Omega \subset R^2$ we denote a bounded polygonal domain with the vertices A_i, $i = 0,\ldots,M$, $A_0 = A_M$, and the boundary $\Gamma = \sum_{i=1}^{M} \bar{\Gamma}_i$ where Γ_i are open straight lines with the end points A_{i-1},A_i. The internal angle of Γ_i and Γ_{i+1} is denoted by ω_i, $i = 1,\ldots,M$, $0 < \omega_i \leq 2\pi$. We mention that we also allow $\omega_i = 2\pi$ and so we include into our consideration the slit domains $(\omega_i = 2\pi)$ when the boundary is two sided (in an obvious sense).

Let $\Gamma^D = \sum_{j\in\mathcal{D}} \bar{\Gamma}_j$ and $\Gamma^N = \Gamma - \Gamma^D = \sum_{j\in\mathcal{N}} \bar{\Gamma}_j$, $\mathcal{D} \cap \mathcal{N} = \emptyset$. We will call Γ^D the Dirichlet boundary and Γ^N the Neumann boundary. Obviously, $\Gamma^D \cup \Gamma^N = \Gamma$.

By $L_2(\Omega) = H^0(\Omega)$ and $H^k(\Omega)$, $k > 0$, integer, we denote the standard Sobolev spaces (with index 2). Also, $H^k_D(\Omega) = H^k(\Omega) \cap H^1_D(\Omega)$ where $H^1_D(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma^D\}$. For $k > 0$ not an integer, we define $H^k(\Omega)$, $H^k_D(\Omega)$ as the usual interpolation space (by the K-method, see [6]):

$$H^{\ell+\theta}(\Omega) = (H^{\ell}(\Omega),H^{\ell+1}(\Omega))_{\theta,q}$$

with $q = 2$, $0 < \theta < 1$, $\ell + \theta = k$. For $k > 1$ we define $H^{k+\theta}_D(\Omega) = H^{k+\theta}(\Omega) \cap H^1_D(\Omega)$. In [3] we have shown that if $\omega_i < 2\pi$ thus

$$H^{k+\theta}_D(\Omega) = (H^k_D(\Omega),H^{k+1}_D(\Omega))_{\theta,q}.$$

Later we will also use $q = \infty$ and will explicitly mention this case.

We will also deal with the Sobolev spaces $H^k(\Gamma_i)$, $H^k(I)$, $I = (a,b)$ which are defined for k integer in the analogous way.

The spaces $H^k(\Omega)$, $H^k_D(\Omega)$, $H^k(\Gamma_i)$, etc., are Hilbert spaces and their inner products will be denoted by $(\cdot,\cdot)_{H^k(\Omega)}$, etc.

For $\kappa > 0$ we let

$$(2.1) \qquad R(\kappa) = \{(x_1,x_2) \mid |x_1| < \kappa, |x_2| < \kappa\}$$

and by $H^k_{PER}(R(\kappa)) \subset H^k(R(\kappa))$ we denote the space of all periodic functions with period 2κ.

By $\mathcal{P}_p(\Omega)$, respectively $\mathcal{T}_p(R(\kappa))$, we denote the space of all algebraic, respectively trigonometric (with period 2κ), polynomials of degree at most p in each variable on Ω, respectively $R(\kappa)$. Analogously we define $\mathcal{P}_p(\Gamma_i)$, $\mathcal{P}_p(I)$, $(I = (a,b))$.

2.2. The model problem and its properties

We will consider the following model problem

$$(2.2) \qquad -\Delta u + u = F \quad \text{on} \quad \Omega$$

$$(2.3a) \qquad u = g \quad \text{on} \quad \Gamma^D$$

$$(2.3b) \qquad \frac{\partial u}{\partial n} = b \quad \text{on} \quad \Gamma^N.$$

The model problem (2.2) (2.3) is a classical case of the elliptic equation problem on a nonsmooth domain. The structure of this problem is well studied. We refer here to [13] and the survey paper [15] where the relevant information and references could be found.

We shall assume that the solution of (2.2), (2.3) can be written in the following form:

$$(2.4) \qquad u = u_1 + u_2 + \sum_{i=1}^{M} u_3^{[i]}$$

where

$$u_1 \in H^q_D(\Omega), \qquad q > 1$$

$$u_2 \in H^k(\Omega), \quad u = g \quad \text{on} \quad \Gamma^D, \qquad k > \frac{3}{2}$$

$$(2.5) \qquad u_3^{[i]} = \sum_{\ell=1}^{n_i} C_\ell^{[i]} \, |\log r_i|^{\gamma_\ell^{[i]}} \, r_i^{\alpha_\ell^{[i]}} \, \Phi_\ell^{[i]}(\Theta_i)\chi^{[i]}(r_i) \in H^1_D(\Omega)$$

with $\alpha_\ell^{[i]} > 0$, $\alpha_{\ell+1}^{[i]} > \alpha_\ell^{[i]}$, $\gamma_\ell^{[i]} > 0$, $\Phi_\ell^{[i]}(\Theta_i)$ and $\chi^{[i]}(r_i)$ are C^∞ (or sufficiently smooth) functions, $\chi^{[i]}(r_i) = 1$ for $0 < r_i < \rho^{[i]} < \frac{1}{4}$, $\chi^{[i]}(r_i) = 0$ for $r_i > 2\rho^{[i]}$. By (r_i, θ_i) we have denoted the polar coordinates with the origin at the vertex A_i of the polygon Ω. The partition (2.5) is typical for the regularity of the solution of the problem (2.2) (2.3). The functions $u_3^{[i]}$ describe the singular behaviour of the solution caused by the corners of Ω or by the abrupt changes of boundary conditions. Function u_2 relates to the nonhomogeneous Dirichlet conditions on Γ^D and u_1 relates to the solution of the problem with the homogeneous Dirichlet conditions. For the details and proofs of the partition (2.5) we refer to [13] [14].

So far we assumed that Ω is a polygon and we considered only the model problem (2.2) (2.3). In Section 6 we will make comments about more general cases.

2.3. The p-version of the finite element method

Let $\bar{\Omega} = \bigcup_{i=1}^{N} \Omega_i$ where Ω_i are (open) triangles or parallelograms. (In Section 6 we will comment on curvilinear triangles and quadrilaterals.) We shall assume that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $\bar{\Omega}_i \cap \bar{\Omega}_j$ is either the empty

set or an entire side or a vertex of Ω_i and Ω_j. We will assume that all vertices of Ω are the vertices of some Ω_j. Ω_j will be called elements.

Denote

$$S_p = \{u \in H^1(\Omega) \mid u \in \mathcal{P}_p(\Omega_i), \quad i = 1,\dots,N\}$$

$$S_p^D = S_p \cap H_D^1(\Omega).$$

Let $\gamma_j \subset \Gamma^D$ be a side of the element Ω_j and let $A_{j_1}, A_{j_2} \in \Gamma^D$ be the end point of γ_j. We define $g_p^{[j]} \in \mathcal{P}_p(\gamma_j)$:

(2.6a) $$g_p^{[j]}(A_{j_i}) = g(A_{j_i}), \qquad i = 1,2$$

(2.6b) $$\int_{\gamma_j} (g_p^{[j]})' h' ds = \int_{\gamma_j} g' h' ds$$

for all $h \in \mathcal{P}_p(\gamma_j)$ with $h(A_{j_i}) = 0$. We define then

(2.6c) $$g_p = \{g_p^{[j]}\}, \qquad \gamma_j \subset \Gamma^D$$

The p-version of the finite element method consists now of finding $u_p \in S_p$, $u_p = g_p$ on Γ^D such that

(2.7) $$(u_p, v)_{H^1(\Omega)} = \int_\Omega Fv \, d\Omega + \int_{\Gamma^D} bv \, ds$$

hold for all $v \in S_p^D$.

(2.7) is the usual definition of the finite element solution when replacing g by g_p on Γ^D so that g_p is the trace of a function in S_p.

Our construction of g_p is slightly restrictive because we assumed

that $g \in H^1(\gamma_j)$ and not $g \in H^k(\gamma_j)$, $\frac{1}{2} < k < 1$. This restriction is not important in practice. It could be avoided at the expense of simplicity of construction of g_p.

Remark. If Ω_i is a parallelogram then $P_p(\Omega)$ is meant as the set of polynomials in variables which are parallel to the sides of Ω_i.

3. APPROXIMATION PROPERTIES OF S_p

Let $Q = (-1,1) \times (-1,1)$; γ_i, $i = 1,2,3,4$ be the sides of Q and γ_5 be the diagonal $x_1 = x_2$ of Q.

LEMMA 3.1. Let $u \in H^k(Q)$. Then there exists a sequence $z_p \in P_p(Q)$, $p = 0,1,2,\ldots$ such that

(3.1) for $k > 0$, $q = 0,1$, $q \leq k$:

$$\|u-z_p\|_{H^q(\Omega)} \leq Cp^{-(k-q)}\|u\|_{H^k(Q)}$$

(3.2) for $k > \frac{3}{2}$:

$$\|u-z_p\|_{H^0(\gamma_i)} \leq Cp^{-(k-1/2)}\|u\|_{H^k(Q)}, \qquad i = 1,\ldots,5$$

(3.3) for $k > \frac{3}{2}$:

$$\|u-z_p\|_{H^1(\gamma_i)} \leq Cp^{-(k-3/2)}\|u\|_{H^k(Q)}, \qquad i = 1,\ldots,5$$

(3.4) for $k > \frac{3}{2}$ and any $x \in Q$:

$$|(u-z_p)(x)| \leq Cp^{-(k-1)}\|u\|_{H^k(Q)}.$$

The constants C in (3.1) - (3.4) depend in general on k but are independent of u and p.

Proof. Let $r_0 > 1$. Then $\bar{Q} \subset R(r_0)$ (see (2.1)). Since Q is a Lipschitz domain, there exists an extension operator T mapping $H^k(Q)$ into $H^k(R(2r_0))$ such that

(3.5a) $$Tu = 0 \quad \text{on} \quad R(2r_0) - R(\tfrac{3}{2} r_0)$$

(3.5b) $$\|Tu\|_{H^k(R(2r_0))} \leq C\|u\|_{H^k(Q)}$$

where C is independent of u. For a concrete construction of T we refer to [3].

Let Φ be the one-to-one mapping of $R(\frac{\pi}{2})$ onto $R(2r_0)$:

$$R(2r_0) \ni x = (x_1,x_2) = \Phi(\xi) = (2r_0 \sin \xi_1,\ 2r_0 \sin \xi_2)$$

with $(\xi_1,\xi_2) = \xi \in R(\frac{\pi}{2})$.

Further, we let

$$\tilde{R} = \Phi^{-1}[R(\tfrac{3}{2} r_0)] \subset R(\tfrac{\pi}{2})$$

where Φ^{-1} denotes the inverse mapping of Φ.

Let $v = Tu$ and

(3.6) $$V(\xi) = v(\Phi(\xi)).$$

Because of (3.5a) we easily see that

(3.7) $$\mathrm{supp}\ V(\xi) \subset \bar{\tilde{R}}.$$

In addition it can be readily seen that

(3.8a) $V(\xi)$ is a periodic function with period 2π

(3.8b) $$\|V(\xi)\|_{H^k(R(\pi))} \leq C\|v\|_{H^k(R(\frac{3}{2} r_0))} \leq C\|u\|_{H^k(Q)}$$

and hence $V \in H^k_{PER}(R(\pi))$.

(3.8c) $V(\xi)$ is a symmetric function with respect to the lines $\xi_i = \pm \frac{\pi}{2}$, $i = 1,2$.

Let us expand the function V in terms of its Fourier series

$$(3.9) \qquad V(\xi_1,\xi_2) = \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_{j\ell} e^{i(j\xi_1+\ell\xi_2)}$$

and denote

$$(3.10) \qquad V_p(\xi_1,\xi_2) = \sum_{j=-p}^{p} \sum_{\ell=-p}^{p} a_{j\ell} e^{i(j\xi_1+\ell\xi_2)}.$$

Obviously, $V_p \in T_p(R(\pi))$.

We have

$$(3.11) \qquad \|V\|^2_{H^k(R(\pi))} \approx \sum_{j,\ell} |a_{j\ell}|^2((1+j^2+\ell^2)^{1/2})^{2k}$$

where $\approx$ has the usual meaning of equivalency. (3.11) yields immediately for $0 \le q \le k$

$$(3.12) \qquad \|V-V_p\|_{H^q(R(\pi))} \le Cp^{-(k-q)}\|V\|_{H^k(R(\pi))}$$

(using (3.8.b))

$$\le Cp^{-(k-q)}\|u\|_{H^k(Q)}$$

with C independent of u.

Let $\hat{\gamma}_i$ $i = 1,\ldots,4$ be the sides of $R(\pi)$ and let $\xi_2 = \hat{\xi}_2$ be one of the sides. Then

$$(3.13) \qquad V(\xi_1,\hat{\xi}_2) - V_p(\xi_1,\hat{\xi}_2) =$$

$$= \Big(\sum_{|j|>p} \sum_{|\ell|\le p} + \sum_{|j|\le p} \sum_{|\ell|>p} + \sum_{|j|>p} \sum_{|\ell|>p}\Big) a_{j\ell} e^{i(j\xi_1+\ell\hat{\xi}_2)}$$

$$= \sum_{|j|>p} b_j^{[1]} e^{ij\xi_1} + \sum_{|j|<p} b_j^{[2]} e^{ij\xi_1} + \sum_{|j|>p} b_j^{[3]} e^{ij\xi_1}$$

where for $|j| > p$:

$$|b_j^{[1]}|^2 = \left(\sum_{|\ell|<p} a_{j\ell} e^{i\ell\hat{\xi}_2}\right)^2 < \left(\sum_{|\ell|<p} |a_{j\ell}|\right)^2 \tag{3.14}$$

(by Schwarz inequality)

$$< \left(\sum_{|\ell|<p} |a_{j\ell}|^2 (1+j^2+\ell^2)^k\right)\left(\sum_{|\ell|<p} (1+j^2+\ell^2)^{-k}\right)$$

$$< A_j\left(\frac{p}{(1+j^2)^k}\right) < CA_j p^{-(2k-1)}$$

where we denote

$$A_j = \sum_{\ell=-\infty}^{\infty} |a_{j\ell}|^2 (1+j^2+\ell^2)^k. \tag{3.15}$$

For $|j| < p$:

$$|b_j^{[2]}|^2 < \left(\sum_{|\ell|>p} |a_{j\ell}|\right)^2 \tag{3.16}$$

$$< A_j \sum_{|\ell|>p} (1+j^2+\ell^2)^{-k}$$

$$< CA_j \int_{p+1}^{\infty} (1+j^2+x^2)^{-k}\, dx$$

$$< CA_j I(k,p)$$

where

$$I(k,p) = \int_{p+1}^{\infty} \frac{dx}{x^{2k}} < Cp^{-(2k-1)} \tag{3.17}$$

provided that $k > 1/2$.

Analogously

$$(3.18)\qquad |b_j^{[3]}|^2 \leq CA_j I(k,p) \leq CA_j p^{-(2k-1)}$$

provided that $k > 1/2$. Hence for $i = 1,2,3,4$,

$$(3.19)\qquad \|V-V_p\|^2_{H^0(\hat{\gamma}_i)}$$

$$\leq C\Big[\sum_{|j|>p} |b_j^{[1]}|^2 + \sum_{|j|\leq p} |b_j^{[2]}|^2 + \sum_{|j|>p} |b_j^{[3]}|^2\Big]$$

$$\leq Cp^{-(2k-1)} \sum_{j=-\infty}^{\infty} A_j$$

$$= Cp^{-(2k-1)} \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} |a_{j\ell}|^2 (1+j^2+\ell^2)^k$$

$$= Cp^{-(2k-1)} \|V\|^2_{H^k(R(\pi))}$$

$$\leq Cp^{-(2k-1)} \|u\|^2_{H^k(Q)}$$

provided that $k > 1/2$.

Now we estimate $\|V-V_p\|_{H^1(\hat{\gamma}_i)}$. We have

$$(3.20)\qquad \frac{d}{d\xi_1}(V-V_p) = i\Big[\sum_{|j|>p} jb_j^{[1]} e^{ij\xi_1} + \sum_{|j|\leq p} jb_j^{[2]} e^{ij\xi_1} + \sum_{|j|>p} jb_j^{[3]} e^{ij\xi_1}\Big]$$

and analogously as in (3.14), for $|j| > p$,

$$(3.21)\qquad |jb_j^{[1]}|^2 \leq \frac{j^2 p}{(1+j^2)^k} A_j .$$

Consider now the function

$$f(x) = \frac{x^2}{(1+x^2)^k} .$$

Then

$$f'(x) = \frac{2x}{(1+x^2)^{k+1}} (1-(k-1)x^2)$$

and hence for $k > 3/2$ and $x^2 > 2$ we obviously get $f'(x) < 0$. Hence $f(x)$ is a decreasing function for $x \geq 2$ and $k > 3/2$.

Hence for $|j| > p$, $p \geq 1$:

$$(3.22)\qquad |jb_j^{[1]}|^2 \leq \frac{p(p+1)^2}{(1+(p+1)^2)^k} A_j \leq CA_j \, p^{-(2k-3)}$$

provided that $k > 3/2$. For $j \leq p$ we get analogously as in (3.16),

$$(3.23)\qquad |jb_j^{[2]}|^2 \leq CA_j \int_{p+1}^{\infty} \frac{j^2 dx}{(1+j^2+x^2)^k} \leq CA_j \, p^{-(2k-3)}$$

provided $k > 3/2$.

Finally, for $j > p$,

$$(3.24)\qquad |jb_j^{[3]}|^2 \leq CA_j \int_{p+1}^{\infty} \frac{j^2 dx}{(1+j^2+x^2)^k}$$

$$= CA_j \left[\int_{p+1}^{\infty} \left(\frac{1}{(1+j^2+x^2)^{k-1}} - \frac{1+x^2}{(1+j^2+x^2)^k} \right) dx \right]$$

$$\leq CA_j \, p^{-(2k-3)}$$

provided that $k > 3/2$. Hence from (3.20) and (3.19)

$$\|V-V_p\|^2_{H^1(\hat{\gamma}_i)} \leq Cp^{-(2k-3)}\|u\|^2_{H^k(Q)}. \tag{3.25}$$

Let us estimate now $|(V-V_p)(\xi_1,\xi_2)|$. Because $V-V_p \in H^k_{PER}(R(\pi))$, we can assume without loss of generality that $(\xi_1,\xi_2) \in \hat{\gamma}_i$. Using (3.13) and (3.22) we get for $k > 3/2$

$$\Big(\sum_{|j|>p} |b_j^{[1]}|\Big)^2 \leq \Big(\sum_{|j|>p} j^2|b_j^{[1]}|^2\Big)\Big(\sum_{|j|>p} \frac{1}{j^2}\Big) \tag{3.26}$$

$$\leq Cp^{-(2k-3)}\|u\|^2_{H^k(Q)} \frac{1}{p} \leq Cp^{-2(k-1)}\|u\|^2_{H^k(Q)}.$$

Using (3.13), (3.16) we get for $k > 3/2$

$$\Big(\sum_{|j|\leq p} b_j^{[2]}\Big)^2 \leq \Big(\sum_{|j|\leq p} |b_j^{[2]}|^2\Big)p \leq Cp^{-2(k-1)}\|u\|^2_{H^k(Q)}. \tag{3.27}$$

Finally, using (3.13), (3.24) we get for $k > 3/2$

$$\Big(\sum_{|j|>p} |b_j^{[3]}|\Big)^2 \leq Cp^{-2(k-1)}\|u\|^2_{H^k(q)}. \tag{3.28}$$

Combining (3.26), (3.27), (3.28) and (3.13) we get for $k > 3/2$

$$|(V-V_p)(\xi_1,\xi_2)| \leq Cp^{-(k-1)}\|u\|_{H^k(Q)}. \tag{3.29}$$

Let us prove now (3.19), (3.25) for $\hat{\gamma}_5$. We have

(3.30) $$V(\xi,\xi) - V_p(\xi,\xi) =$$

$$= \Big(\sum_{|j|>p}\sum_{|\ell|<p} + \sum_{|j|<p}\sum_{|\ell|>p} + \sum_{|j|>p}\sum_{|\ell|>p}\Big) a_{j\ell}\, e^{i(j+\ell)\xi}$$

$$= \sum_{q=-\infty}^{\infty} (C_q^{[1]} + C_q^{[2]} + C_q^{[3]}) e^{iq\xi}$$

where

(3.31a) $$C_q^{[1]} = \sum_{\substack{j+\ell=q \\ |j|>p,\,|\ell|<p}} a_{j\ell}$$

(3.31b) $$C_q^{[2]} = \sum_{\substack{j+\ell=q \\ |j|<p,\,|\ell|>p}} a_{j\ell}$$

(3.31c) $$C_q^{[3]} = \sum_{\substack{j+\ell=q \\ |j|>p,\,|\ell|>p}} a_{j\ell}.$$

Now by the Schwarz inequality

(3.32) $$|C_q^{[1]}|^2 < \Big(\sum_{\substack{j+\ell=p \\ |j|>p,\,|\ell|<p}} |a_{j\ell}|^2 (1+j^2+\ell^2)^k\Big)\Big(\sum_{\substack{j+\ell=q \\ |j|>p,\,|\ell|<p}} (1+j^2+\ell^2)^{-k}\Big)$$

$$< A_q I^{[1]}(k,p,q)$$

where

(3.33) $$A_q = \sum_{j+\ell=q} |a_{j\ell}|^2 (1+j^2+\ell^2)^k$$

$$I^{[1]}(k,p,q) < \frac{1}{(1+p^2)^k} N(p,q)$$

and $N(p,q)$ is the number of terms in the second term on the right hand side of (3.32). Obviously, $N < 2p$. Hence

$$|C_q^{[1]}|^2 < CA_q\, p^{-(2k-1)}. \tag{3.34}$$

Analogously,

$$|C_q^{[2]}|^2 < CA_q\, p^{-(2k-1)}. \tag{3.35}$$

Finally,

$$|C_q^{[3]}|^2 < A_q\Big(\sum_{\substack{j+\ell=q\\ |j|>p,\,|\ell|>p}} (1+j^2+\ell^2)^{-k}\Big) \tag{3.36}$$

$$< CA_q \int_{p+1}^{\infty} \frac{dx}{x^{2k}} < Cp^{-(2k-1)}A_q.$$

provided that $k > 1/2$. From (3.33) we see that

$$\sum_{q=-\infty}^{\infty} A_q < C\|V\|^2_{H^k(R(\pi))} < C\|u\|^2_{H^k(Q)}. \tag{3.37}$$

(3.37) together with (3.34), (3.35), (3.36) and (3.30) yields

$$\|V-V_p\|_{H^0(\hat{\gamma}_5)} < Cp^{-(k-1/2)}\|u\|_{H^k(Q)}. \tag{3.38}$$

Now we estimate $\|V-V_p\|_{H^1(\hat{\gamma}_5)}$. Using (3.30) we have

$$\frac{d}{d\xi}(V-V_p)(\xi,\xi) = \sum_{q=-\infty}^{\infty} iq(C_q^{[1]} + C_q^{[2]} + C_q^{[3]})e^{iq\xi} \tag{3.39}$$

and

$$|iqC_q^{[1]}|^2 \leq A_q \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|\leq p}} \frac{q^2}{(1+j^2+\ell^2)^k}.$$

If $p^2 \geq q^2$ then

$$\sum_{\substack{j+\ell=q \\ |j|>p,|\ell|\leq p}} \frac{q^2}{(1+j^2+\ell^2)^k} \leq p^2 \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|\leq p}} \frac{1}{(1+j^2+\ell^2)^k}$$

$$\leq C \frac{p^3}{(1+p^2)^k} \leq Cp^{-(2k-3)}.$$

If $p^2 < q^2$ then

$$p^2 < q^2 = (j+\ell)^2 \leq 2(j^2+\ell^2)$$

and we get

$$\sum_{\substack{j+\ell=q \\ |j|>p,|\ell|\leq p}} \frac{q^2}{(1+j^2+\ell^2)^k} \leq \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|\leq p}} \frac{q^2}{(1+\frac{q^2}{2})^k} \leq (2p)\left(\frac{C}{q^{2k-2}}\right)$$

$$\leq Cp^{-(2k-3)} \qquad \text{provided } k > 1.$$

Hence,

$$|qC_q^{[1]}|^2 \leq CA_q\, p^{-(2k-3)}. \tag{3.40}$$

Similarly,

$$|qC_q^{[2]}|^2 \leq CA_q\, p^{-(2k-3)}. \tag{3.41}$$

Finally,

$$|qC_q^{[3]}|^2 \leq CA_q \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|>p}} \frac{q^2}{(1+j^2+\ell^2)^k}. \tag{3.42}$$

For $q^2 < p^2$ we get

$$(3.43) \qquad \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|>p}} \frac{q^2}{(1+j^2+\ell^2)^k} < p^2 \sum_{|j|>p} \frac{1}{(1+j^2)^k} < Cp^{-(2k-3)}.$$

If $q^2 > p^2$ then

$$(3.44) \qquad \sum_{\substack{j+\ell=q \\ |j|>p,|\ell|>p}} \frac{q^2}{(1+j^2+\ell^2)^k} < \sum_{q=p+1}^{\infty} \frac{q^2}{\left(1+\frac{q^2}{2}\right)^k} < Cp^{-(2k-3)}$$

and hence

$$(3.45) \qquad |qC_q^{[3]}|^2 < CA_q \, p^{-(2k-3)}.$$

Combining (3.40), (3.41), (3.45) and (3.39), (3.38), we get

$$(3.46) \qquad \|V-V_p\|_{H^1(\hat{\gamma}_5)} < Cp^{-(k-3/2)}\|u\|_{H^k(Q)}.$$

Because of (3.8c), $V_p(\Phi^{-1}(x)) \in P_p(Q)$. Further, Φ is a regular mapping of $R(r_0)$ on Q, $(r_0 < \frac{\pi}{2})$ and $\Phi(\hat{\gamma}_i) \supset \gamma_i$. Hence for k integer (3.1) follows immediately from (3.12), (3.2) from (3.19) and (3.38), (3.3) from (3.25), (3.46), and (3.4) from (3.29).

There is a known one dimensional version of Lemma (3.1). For the proof see e.g. [9]

LEMMA 3.2. Let $I = (-1,1)$, $u \in H^k(I)$, $k > 1$. Then there exists a polynomial $z_p \in P_p(I)$ such that

$$(3.47) \qquad u(\pm 1) = z_p(\pm 1)$$

and for $t = 0,1$:

(3.48) $$\|u-z_p\|_{H^t(I)} \leq Cp^{-(k-t)}\|u\|_{H^k(I)}.$$

Let us mention another form of the approximation theorem

LEMMA 3.3. Let $u \in H^1(I)$

$$\int_{-1}^{+1} (1-x^2)^k \left(\frac{d^{k+1}u}{dx^{k+1}}\right)^2 dx = A^2, \qquad k = 0,1,\ldots$$

Then there exists a polynomial $z_p \in P_p(I)$ such that

(3.49) $$u(\pm 1) = z_p(\pm 1)$$

and for $t = 0,1$ and $p > k$:

(3.50) $$\|u-z_p\|_{H^t(I)} \leq Cp^{-(k-t+1)} A$$

where C depends on k but not on A and p.

For the proof see [5] or [9].

4. THE CONVERGENCE RATE OF THE p-VERSION: THE CASE WHEN $u \in H^k(\Omega)$

4.1. The case $k > 3/2$.

In this section we will analyze the rate of convergence of the p-version in the case that the exact solution $u \in H^k(\Omega)$, $k > 3/2$.

THEOREM 4.1. Let $u \in H^k(\Omega)$, $k > 3/2$ be the solution of (2.2) (2.3). Then there exists $u_p \in S_{p+1}$, $u_p = g_p$ on Γ^D (see 2.6a,b,c) such that

$$(4.1) \qquad \|u-u_p\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}$$

where C depends on the partition of Ω and on k, but is independent of u and p.

Proof. Let Ω_i, $i = 1,\dots,N$ be the elements of the partition of Ω. First let us construct the functions $z_p^{[i]}$ as in Lemma 3.1. The lemma is applicable because a linear transformation maps the parallelogram or triangular element on a square or on a right angled triangle, preserving the polynomials. Hence

$$(4.2) \qquad \|u-z_p^{[i]}\|_{H^1(\Omega_i)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega_i)}.$$

Using (3.4) we can assume that $u = z_p^{[i]}$ at the vertices of Ω_i by adding a linear (triangle) or bilinear (parallelogram) function to $z_p^{[i]}$.

Let now $\gamma = \bar{\Omega}_j \cap \bar{\Omega}_\ell$ and A_1, A_2 be the end points of γ. Now $z_p^{[j]} \neq z_p^{[\ell]}$ on γ. Denote $w_{j\ell} = z_p^{[j]} - z_p^{[\ell]}$ on γ. Then because $z_p^{[j]}(A_i) = u(A_i) = z_p^{[\ell]}(A_i)$, we have $w_{j\ell}(A_i) = 0$. Using (3.2) and (3.3) we get for $t = 0,1$

$$(4.3) \qquad \|w_{j\ell}\|_{H^t(\gamma)} \leq Cp^{-(k-1/2-t)}\|u\|_{H^k(\Omega)}.$$

If Ω_j is a parallelogram, then we can assume without any loss of generality that $\Omega_j = Q = (-1,1) \times (-1,1)$ and $\gamma = \{x \mid |x_1| \leq 1, \; x_2 = -1\}$.

Let

$$(4.4) \qquad \phi_p(x_2) = \frac{e^{-p(x_2+1)} - e^{-2p}}{1-e^{-2p}}, \qquad x_2 \in I = (-1,1)$$

Then

$$(4.5a) \qquad \|\phi_p\|_{H^0(I)} \leq Cp^{-1/2}$$

$$(4.5b) \qquad \|\phi_p\|_{H^1(I)} \leq Cp^{1/2}$$

and

$$(4.6) \qquad \phi_p(-1) = 1, \qquad \phi_p(1) = 0.$$

Using Lemma 3.3 with $k = 1$ and $t = 0,1$, there is a $\psi_p(x_2) \in \mathcal{P}_p(I)$ such that

$$(4.7) \qquad \|\psi_p - \phi_p\|_{H^t} \leq Cp^{-2+t}\Big[\int_I p^4 e^{-2p(x_2+1)}(1-x_2^2)dx_2\Big]^{1/2}$$

$$\leq Cp^t\Big(\int_0^2 ye^{-2py}dy\Big)^{1/2} \leq Cp^{t-1}.$$

Hence, for

$$\xi_{j\ell} = w_{j\ell}\psi_p(x_2) \in \mathcal{P}_p(\Omega_j)$$

we get

$$\xi_{j\ell} = 0 \quad \text{on} \quad \partial\Omega_j - \gamma$$

$$\xi_{j\ell} = w_{j\ell} \quad \text{on} \quad \gamma$$

and

$$(4.8) \qquad \|\xi_{j\ell}\|_{H^1(\Omega_j)} \le C[\|w_{j\ell}\|_{H^1(\gamma)}\|\psi_p\|_{H^0(I)} + \|w_{j\ell}\|_{H^0(\gamma)}\|\psi_p\|_{H^1(I)}]$$

$$\le C[\|w_{j\ell}\|_{H^1(\gamma)}(\|\phi_p\|_{H^0(I)} + \|\phi_p-\psi_p\|_{H^0(I)})$$

$$+ \|w_{j\ell}\|_{H^0(\gamma)}(\|\phi_p\|_{H^1(I)} + \|\phi_p-\psi_p\|_{H^1(I)})]$$

by (4.3)

$$\le C[p^{-(k-3/2)}(p^{-1/2}+p^{-1}) + p^{-(k-1/2)}(p^{1/2}+1)]\|u\|_{H^k(\Omega)}$$

$$\le Cp^{-(k-1)}\|u\|_{H^k(\Omega)}.$$

Repeating this process for all four sides of Ω_j we can adjust $z_p^{[j]}$ so that *the continuity across* γ *is obtained* and (4.2) still holds on Ω_j.

So far we assumed that Ω_j was a rectangle. Now let Ω_j be a triangle. Then without any loss of generality we can assume that

$$\Omega_j = \{x \mid 0 < x_1 < 1, \quad 0 < x_2 < x_1\}$$

$$\gamma = \{x \mid 0 \le x_1 \le 1, \quad x_2 = 0\}$$

and we assume that $w_{j\ell}(0) = w_{j\ell}(1) = 0$.

Let now

$$\xi_{j\ell}(x_1,x_2) = \psi_p(2x_2-1)[(x_1-x_2)w_{j\ell}(x_1) + (1-x_1)w_{j\ell}(x_1-x_2)].$$

Obviously, $\xi_{j\ell}$ is a polynomial of degree at most $p+1$ in each variable, which vanishes on $\partial\Omega_j - \gamma$ (because $w_{j\ell}(0) = w_{j,\ell}(1) = 0$) and $\xi_{j\ell}(x_1,0) = w_{j,\ell}(x_1)$. Now by quite similar arguments as before we see that

$$\|\xi_{j\ell}\|_{H^1(\Omega_j)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)} .$$

Adjusting $z_p^{[j]}$ by $\xi_{j\ell}$ we achieve the continuity across γ. The continuity across each side γ of Ω_i belonging to Γ^D can be similarly obtained. This completes the proof for $g = 0$ and for g a polynomial of degree $p_0 < p$ on every $\partial\Omega_j \subset \Gamma^D$.

If g is general we can proceed quite analogously. Let $\gamma \subset \Gamma^D$ be the side of the element Ω_j with the end points A_i, $i = 1,2$. By Lemma 3.1 we have for $t = 0,1$ and $k > 3/2$:

$$\|z_p^{[j]}-u\|_{H^t(\gamma)} \leq Cp^{-(k-t-1/2)}\|u\|_{H^k(\Omega)} .$$

On the other hand, by the imbedding theorem we have for $s > 1$

$$\|g\|_{H^s(\gamma)} = \|u\|_{H^s(\gamma)} \leq C\|u\|_{H^{s+1/2}(\Omega)} .$$

Applying Lemma 3.2 we have for $t = 0,1$:

$$\|g-g_p^{[j]}\|_{H^t(\gamma)} \leq Cp^{-(k-t)}\|g\|_{H^k(\gamma)} \leq Cp^{-(k-t)}\|u\|_{H^{k+1/2}(\Omega)} .$$

Hence,

$$\|z_p^{[j]}-g_p^{[j]}\|_{H^t(\gamma)} \leq Cp^{-(k-t-1/2)}\|u\|_{H^k(\Omega)} \tag{4.9}$$

and

$$z_p^{[j]}(A_i) = g_p^{[j]}(A_i)$$

so that we can construct the adjustment of $z_p^{[j]}$ exactly as before.

This completes the proof.

4.2. The case $1 < k < 3/2$

We assumed in Section 4.1 that $u \in H^k$, $k > 3/2$. Let us analyze now the general case.

THEOREM 4.2. Let $u \in H^k(\Omega)$, $k > 1$, be the solution of (2.2) (2.3) such that

$$u = u_1 + u_2$$

$$u_1 \in H_D^{k_1}(\Omega), \qquad u_2 \in H^{k_2}(\Omega), \qquad\qquad k_2 > 3/2$$

and such that if $k_1 < 3/2$ then Ω is a Lipschitz domain (i.e. $\omega_i < 2\pi$). Then there exists $u_p \in S_{p+1}$, $u_p = g_p$ on Γ^D (see 2.6a,b,c) such that

$$\|u-u_p\|_{H^1(\Omega)} \le Cp^{-(k-1)}\|u\|_{H^k(\Omega)}, \qquad k = \min(k_1,k_2) \tag{4.10}$$

Proof. Because of Theorem 4.1 we can assume that $u_2 = 0$, i.e. $g = 0$ on Γ^D and that $1 < k < 2$. Let us assume first that for any $0 < t < \infty$ we can write

$$u = v_1(t) + v_2(t) \tag{4.11}$$

where $v_i \in H_D^1(\Omega)$, $i = 1,2$ and

$$\|v_1\|_{H^1(\Omega)} \le Ct^{k-1}\|u\|_{H^k(\Omega)} \tag{4.12a}$$

$$\|v^2\|_{H^2(\Omega)} \le Ct^{k-2}\|u\|_{H^k(\Omega)} \tag{4.12b}$$

with C independent of u. Then by Theorem 4.1 there exists $z_p \in S_p^D$ such that

$$\|z_p-v_2\|_{H^1(\Omega)} \le Cp^{-1}\|v_2\|_{H^2(\Omega)} \le Cp^{-1}t^{k-2}\|u\|_{H^k(\Omega)}.$$

Choosing $t = 1/p$, (4.12a) gives

$$\|v_1\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}.$$

We get by the triangle inequality

$$\|z_p - u\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)} \tag{4.13}$$

i.e. the estimate (4.1).

The assumed splitting (4.11), (4.12) is equivalent to the definition of the interpolated space $(H^1_D(\Omega), H^2(\Omega) \cap H^1_D(\Omega))_{\theta,\infty}$ defined by the K-method (see [6]).

We have shown in [3] that

$$\begin{aligned}
&(H^1_D(\Omega), H^2(\Omega) \cap H^1_D(\Omega))_{\theta,\infty} = \\
&= (H^1(\Omega), H^2(\Omega))_{\theta,\infty} \cap H^1_D(\Omega) \\
&= B^{1+\theta}_{2,\infty}(\Omega) \cap H^1_D(\Omega) \supset H^k_D \\
&= (H^1(\Omega), H^2(\Omega))_{\theta,2} \cap H^1_D(\Omega) \\
&= H^k(\Omega) \cap H^1_D(\Omega)
\end{aligned}$$

where $B^{1+\theta}_{2,\infty}$ is the usual Besov space. Hence (4.11), (4.12) hold if $u \in H^k_D(\Omega)$ and Theorem 4.2 is proven.

Remark 4.1. We mention that we have proven slightly more then the claim of Theorem 4.1, namely that we can use the Besov space $B^k_{2,\infty}(\Omega)$ instead of $H^k(\Omega)$. This of course follows easily from the interpolation theory.

Remark 4.2. The assumption $\omega_i < 2\pi$ (i.e. the exclusion of the slit domain) was made here only because it is used in a result from [3] quoted by us.

4.3. The rate of convergence of the p-version of the finite element method

THEOREM 4.3. Let $u \in H^k(\Omega)$, $k > 1$ be the solution of (2.2), (2.3). Assume further that g is such that

$$u = u_1 + u_2$$

$$u_1 \in H_D^{k_1}(\Omega), \qquad u_2 \in H^{k_2}(\Omega), \qquad k_2 > 3/2$$

and that Ω is a Lipschitz domain if $k_1 < 3/2$. Let u_p be the finite element solution based on the p-version defined in Section 2. Then

$$\|u-u_p\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}, \qquad k = \min(k_1,k_2). \tag{4.14}$$

The constant C is independent of p and u. It depends on the factorization of Ω into elements Ω_i.

Proof. If $g = 0$, then (4.14) follows immediately from Theorem 4.2 because

$$\|u-u_p\|_{H^1(\Omega)} \leq C\|u-z_p\|_{H^1(\Omega)}.$$

If $g \neq 0$ then denote by U_p the exact solution of the problem (2.2), (2.3) when replacing g by g_p. Denoting $\omega = u - U_p$, function ω obviously satisfies

$$-\Delta\omega + \omega = 0$$

$$\frac{\partial\omega}{\partial n} = 0 \text{ on } \Gamma^N$$

$$\omega = g - g_p \quad \text{on} \quad \Gamma^D.$$

Using Lemma 3.2 and the same argument as in Theorem 4.1, i.e. the extension by the function ϕ_p in (4.4) we conclude that there is a function $w \in H^1(\Omega)$ such that

$$\|w\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}$$

and

$$w = \omega = g - g_p \quad \text{on} \quad \Gamma^D.$$

Hence

$$\|\omega\|_{H^1(\Omega)} \leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}, \tag{4.15}$$

because ω minimizes $\|\cdot\|_{H^1(\Omega)}$ among all functions with trace $g - g_p$ on Γ^D. By Theorem 4.2 and a basic property of the Finite Element Method, we have

$$\begin{aligned}\|u_p - U_p\|_{H^1(\Omega)} &\leq C\|z_p - U_p\|_{H^1(\Omega)} \\ &\leq C(\|z_p - u\|_{H^1(\Omega)} + \|u - U_p\|_{H^1(\Omega)}) \\ &\leq Cp^{-(k-1)}\|u\|_{H^k(\Omega)}\end{aligned}$$

and Theorem 4.3 is proven.

5. THE CONVERGENCE RATE OF THE p-VERSION: THE CASE OF THE SINGULAR SOLUTION

In Section 4 we analyzed the rate of convergence of the p-version under the assumption that $u_3^{[i]} = 0$ in (2.4). In this section we will consider in detail the case when $u = u_3^{[i]}$.

5.1. An approximation result

Let $Q = (-1,1) \times (-1,1)$ as in Section 3. Denote $\tilde{x}_i = x_i + 1$, $i = 1,2$ and let for $\kappa > 1$, $0 < \rho < 1$,

$$S_\kappa = \{x \in Q \mid \frac{1}{\kappa}\tilde{x}_1 < \tilde{x}_2 < \kappa\tilde{x}_1\}$$

$$S_\kappa^\rho = S_\kappa \cap \{x \mid \tilde{x}_1^2 + \tilde{x}_2^2 < \rho^2\}$$

$$Q_0 = \{x \mid 0 < \tilde{x}_1 < 1, \quad 0 < \tilde{x}_2 < 1\}$$

$$\tilde{Q}_0 = \{x \mid 0 < \tilde{x}_1 < 1/2, \quad 0 < \tilde{x}_2 < 1/2\}$$

$$S_0^{\kappa,\rho} = S_\kappa^\rho \cap Q_0$$

$$R_\kappa = S_\kappa \cap Q_0$$

$$\tilde{R}_\kappa = S_\kappa \cap \tilde{Q}_0.$$

Let $\kappa_0 > \kappa > 1$. Fig. 5.1 shows the domains under consideration.

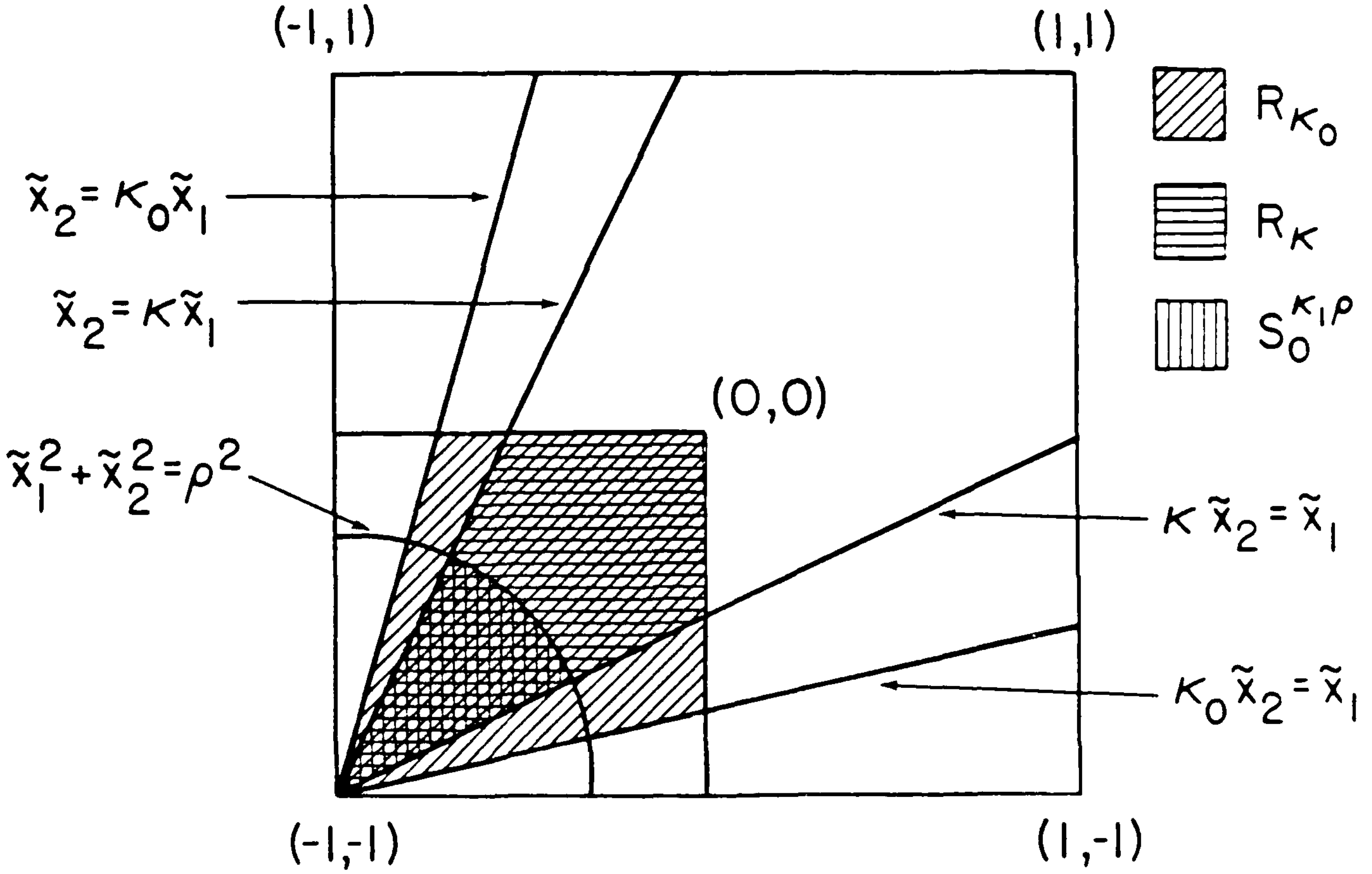

Fig. 5.1

Let (r,θ) be the polar coordinates with origin $(-1,-1)$; $r^2 = \tilde{x}_1^2 + \tilde{x}_2^2$, $\theta = \arctan(\frac{\tilde{x}_2}{\tilde{x}_1})$.

Let

$$(5.1) \qquad \xi(\tilde{x}_1,\tilde{x}_2) = (\tilde{x}_1 - \kappa\tilde{x}_2)(\kappa\tilde{x}_1 - \tilde{x}_2) = r^2\phi_1(\theta).$$

Obviously $\phi_1(\theta)$ is an analytic function in θ, and ξ is a polynomial which vanishes on the lines $\tilde{x}_1 = \kappa\tilde{x}_2$ and $\tilde{x}_1 = \frac{1}{\kappa}\tilde{x}_2$.

Let, for $\alpha > 0$, $\gamma > 0$

$$u(\tilde{x}_1,\tilde{x}_2) = r^{\alpha}|\log r|^{\gamma}\chi(r)\Phi(\theta) \tag{5.2}$$

where $\Phi(\theta)$, $\chi(r)$ are sufficiently smooth functions (e.g. C^∞ functions) and

$$\chi(r) = 1 \quad \text{for } 0 \leq r \leq \frac{\rho}{3}$$

$$\chi(r) = 0 \quad \text{for } \frac{2\rho}{3} \leq r, \; 0 < \rho < \frac{1}{2}$$

is a function defined on Q. We shall assume that u vanishes on the lines $\tilde{x}_1 = \kappa\tilde{x}_2$ and $\tilde{x}_1 = \frac{1}{\kappa}\tilde{x}_2$, and has support in R_{κ_0}. Then

$$u_0(\tilde{x}_1,\tilde{x}_2) = \frac{u(\tilde{x}_1,\tilde{x}_2)}{\xi(\tilde{x}_1,\tilde{x}_2)} = r^{\alpha-2}|\log r|^{\gamma}\chi(r)\psi(\theta) \tag{5.3}$$

where $\psi(\theta)$ is once more smooth (e.g. C^∞ function).

Now we can write

$$u(\tilde{x}_1,\tilde{x}_2) = \xi(\tilde{x}_1,\tilde{x}_2)u_0(\tilde{x}_1,\tilde{x}_2). \tag{5.4}$$

The main result of this section is

THEOREM 5.1. Let u be given by (5.2). Then there exists $z_p \in P_{p+2}(Q)$ such that $z_p = 0$ on the lines $\tilde{x}_1 = \kappa\tilde{x}_2$ and $\tilde{x}_1 = \frac{1}{\kappa}\tilde{x}_2$, and for $\kappa_0 > \kappa$,

$$\|u-z_p\|_{H^1(\tilde{R}_{\kappa_0})} \leq C|\log p|^{\gamma}p^{-2\alpha} \tag{5.5}$$

where C is a constant independent of p.

We will need a series of lemmas to prove Theorem 5.1.

Let $\omega(r)$, $0 \leq r < \infty$ be a C^∞ function satisfying

$$\omega(r) = 0 \quad \text{for} \quad 0 \le r \le 1$$

$$\omega(r) = 1 \quad \text{for} \quad 2 \le r < \infty.$$

Further, for any $\Delta > 0$ let

$$\omega^{\Delta}(r) = \omega(\tfrac{r}{\Delta}) \tag{5.6}$$

and

$$v = \omega^{\Delta}u_0 \tag{5.7a}$$

$$w = (1-\omega^{\Delta})u_0. \tag{5.7b}$$

Then obviously

$$u_0 = v + w. \tag{5.8}$$

It can be readily seen that

$$v = 0 \quad \text{for} \quad 0 \le r \le \Delta$$

$$w = 0 \quad \text{for} \quad r \ge 2\Delta.$$

LEMMA 5.1. Let $k = k_1 + k_2$ where $0 \le k_1, \ k_2 \le k$ are integers. Then there exists a constant $C(k)$ such that for $x = (x_1,x_2) \in R_{\kappa_0}$

$$\left|\frac{\partial^k v}{\partial x_1^{k_1} \partial x_2^{k_2}}\right| \begin{cases} \le C(k)|\log \Delta|^{\gamma}(1+x_1)^{\alpha-2-k} & \text{on } R_\kappa \\ = 0 & \text{on } S_\kappa^{\Delta}. \end{cases} \tag{5.9}$$

<u>Proof</u>. We have

$$\left|\frac{\partial v}{\partial r}\right| \le C[|r^{\alpha-3}|\log r|^{\gamma}\chi(r)\psi(\theta)\omega^{\Delta}(r)| + |r^{\alpha-3}|\log r|^{\gamma-1}\chi(r)\psi(\theta)\omega^{\Delta}(r)|$$

$$+ |r^{\alpha-2}|\log r|^{\gamma}\chi'(r)\psi(\Theta)\omega^{\Delta}(r)| + |r^{\alpha-2}|\log r|^{\gamma}\chi(r)\psi(\Theta)\omega^{\Delta\prime}(r)\,\frac{1}{\Delta}|].$$

Note that the third term on the right hand side is 0 except for $\frac{\rho}{3} < r < \frac{2\rho}{3}$ and the fourth term is zero for $r > 2\Delta$. Hence for $\Delta < \frac{\rho}{6}$ we have

$$\begin{aligned}
\left|\frac{\partial v}{\partial r}\right| &= 0 && \text{for } 0 < r < \Delta \\
&< C|\log \Delta|^{\gamma} r^{\alpha-3} && \text{for } \Delta < r < 2\Delta \\
&< C|\log \Delta|^{\gamma} \max(r^{\alpha-2}, r^{\alpha-3}) && \text{for } r > 2\Delta,\ \frac{\rho}{3} < r < \frac{2\rho}{3} \\
&< C|\log \Delta|^{\gamma} r^{\alpha-3} && \text{for } r > 2\Delta,\ r \notin (\frac{\rho}{3}, \frac{2\rho}{2})
\end{aligned}$$

Hence

$$\left|\frac{\partial v}{\partial r}\right| < C|\log \Delta|^{\gamma} r^{\alpha-3}$$

for all r.

This process can be repeated to obtain

$$\left|\frac{\partial^k v}{\partial r^k}\right| < C(k)|\log \Delta|^{\gamma} r^{\alpha-2-k}. \tag{5.10}$$

In S_{κ_0} we have

$$D_1(1+x_1) < r < (1+x_1)D_2$$

$$\frac{\partial r}{\partial x_1} = \cos\Theta, \qquad \frac{\partial r}{\partial x_2} = \sin\Theta$$

$$\frac{\partial \Theta}{\partial x_1} = \frac{\sin\Theta}{r}, \qquad \frac{\partial \Theta}{\partial x_2} = \frac{\cos\Theta}{r}$$

and $\psi(\Theta)$ is smooth. Hence (5.10) gives immediately (5.9).

In what follows we will assume that v satisfies (5.9) and not the explicit form (5.3), (5.7a).

Let

$$v(x_1,x_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} P_i(x_1) P_j(x_2) \tag{5.11}$$

where $P_i(x_\ell) = P_i(x_\ell,\beta,\beta)$, $\beta > -\frac{1}{2}$ are Jacobi polynomials of index β which will be determined later. Then

$$a_{ij} = C_i C_j (i+1)(j+1) \int_{-1}^{+1} \int_{-1}^{+1} v(x_1,x_2) P_i(x_1) P_j(x_2) (1-x_1^2)^\beta (1-x_2^2)^\beta dx_1 dx_2$$

where C_i, C_j are bounded from above and below independently of i, j but depending on β (see [12], p. 841, formula 7.391.1).

Define

$$v_p(x_1,x_2) = \sum_{i=0}^{p} \sum_{j=0}^{p} a_{ij} P_i(x_1) P_j(x_2), \tag{5.12}$$

$$b_i(x_2) = \sum_{j=0}^{\infty} a_{ij} P_j(x_2), \tag{5.13a}$$

$$b_i^{[p]}(x_2) = \sum_{j=0}^{p} a_{ij} P_j(x_2) \tag{5.13b}$$

with

$$b_i(x_2) = C_i (i+1) \int_{-1}^{+1} v(x_1,x_2)(1-x_1^2)^\beta P_i(x_1) dx_1. \tag{5.14}$$

It can be readily seen that

$$v = \sum_{i=0}^{\infty} b_i(x_2) P_i(x_1) \tag{5.15a}$$

(5.15b) $$v_p = \sum_{i=0}^{p} b_i^{[p]}(x_2)P_i(x_1).$$

Let

(5.16) $$\psi_p(x_1,x_2) = \sum_{i=0}^{p} b_i(x_2)P_i(x_1),$$

then

(5.17) $$v - v_p = (v-\psi_p) + (\psi_p - v_p) = \sigma_p + \rho_p.$$

We now mention a lemma which will be needed.

LEMMA 5.2 [Bernstein]. Let $\beta > -\frac{1}{2}$, then

(5.18) $$|P_i(x,\beta,\beta)| \le \frac{C}{i^{1/2}}(1-x^2)^{-(\frac{\beta}{2}+\frac{1}{4})}.$$

For the proof, see [10], p. 299.

LEMMA 5.3. Let $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$. Then

(5.19) $$\left|\frac{d^m b_i(x_2)}{dx_2^m}\right| \le C(i+1)^{1/2}|\log \Delta|^{\gamma}(1+x_2)^{\alpha-m+\frac{\beta}{2}-\frac{5}{4}}, \qquad i = 0,1,2,...$$

where C is independent of i, x_2 but depends on α, β, γ, m.

<u>Proof</u>. Using (5.14) and (5.18) we get for $i \ge 1$

$$\left|\frac{d^m b_i(x_2)}{dx_2^m}\right| = \left|C_i(i+1)\int_{-1}^{+1}\frac{d^m v}{dx_2^m}(1-x_1^2)^{\beta}P_i(x_1)dx_1\right|$$

$$\le C_i(i+1)^{1/2}\int_{-1}^{+1}\left|\frac{d^m v}{dx_2^m}\right|(1-x_1^2)^{\beta}(1-x_1^2)^{-(\frac{\beta}{2}+\frac{1}{4})}dx_1$$

$\dfrac{d^m v}{dx_2^m}$ has support in R_{κ_0} and is zero for x_1 lying outside the interval

$$I_1(x_2) = [-1 + \frac{1}{\kappa_0}(1+x_2),\ -1 + \kappa_0 (1+x_2)].$$

Using Lemma 5.1 we get

$$\left|\frac{d^m b_i(x_2)}{dx_2^m}\right| < C(i+1)^{1/2} \int_{I_1(x_2)} |\log \Delta|^\gamma (1+x_1)^{\alpha-2-m+\frac{\beta}{2}-\frac{1}{4}} dx_1$$

$$< Ci^{1/2} |\log \Delta|^\gamma (1+x_2)^{\alpha-2-m+\frac{\beta}{2}-\frac{1}{4}+1}$$

provided that $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$ which is (5.19) for $i > 1$. The case $i = 0$ is verified separately (using the fact that $\beta > -1/2$).

LEMMA 5.4. Let $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$. Then

$$\left|\frac{d^m b_i(x_2)}{dx_2^m}\right| < C\,\frac{|\log \Delta|^\gamma}{(i+1)^{1/2}}\,(1+x_2)^{\alpha-m+\frac{\beta}{2}-\frac{7}{4}}. \tag{5.20}$$

Proof. We have (see [12], p. 1039, Formula 8.964)

$$(1-x_1^2)P_i''(x_1) - 2x_1(\beta+1)P_i'(x_1) + i(i+2\beta+1)P_i(x_1) = 0. \tag{5.21}$$

Multiplying (5.21) by $(1-x_1^2)^\beta$ we get

$$-i(i+2\beta+1)(1-x_1^2)^\beta P_i(x) = \frac{d}{dx_1}((1-x_1^2)^{1+\beta}P_i'(x_1)). \tag{5.22}$$

Hence, differentiating (5.14) m times, using (5.22) and integrating by parts we get

$$\left|\frac{d^m b_i(x_2)}{dx_2^m}\right| < C\,\frac{1}{(2\beta+1+i)}\left|\int_{-1}^{+1} \frac{\partial^m v}{\partial x_2^m}\,\frac{d}{dx_1}\left((1-x_1^2)^{1+\beta} P_i'(x_1)\right)dx_1\right| \tag{5.23}$$

$$< \frac{C}{i+1}\int_{-1}^{+1} \left|\frac{\partial}{\partial x_1}\left(\frac{\partial^{m+1} v}{\partial x_1 \partial x_2^m}(1-x_1^2)^{1+\beta}\right)\right| \, |P_i(x_1)|dx_1 .$$

As in the proof of the previous lemma, the integrand is zero outside of $I_1(x_2)$. Further,

$$(5.24)\qquad \left|\frac{\partial}{\partial x_1}\left(\frac{\partial^{m+1} v}{\partial x_1 \partial x_2^m}(1-x_1^2)^{1+\beta}\right)\right|$$

$$< \left|(1-x_1^2)^{1+\beta}\frac{\partial^{m+2} v}{\partial x_1^2 \partial x_2^m}\right| + \left|2(1+\beta)\frac{\partial^{m+1} v}{\partial x_1 \partial x_2^m}(1-x_1^2)^{\beta} x_1\right|$$

$$< C(m)(1+x_1)^{\alpha-m+\beta-3}|\log \Delta|^{\gamma}$$

where we used Lemma 5.1 and the fact that $(1-x_1^2) < 2(1+x_1)$ on $I_1(x_2)$. Hence by Lemma 5.2, (5.23), (5.24) we get

$$(5.25)\qquad \left|\frac{d^m b_i(x_2)}{dx_2^m}\right| < C\frac{|\log \Delta|^{\gamma}}{(i+1)^{3/2}}\int_{I_1(x_2)}(1+x_1)^{\alpha-m+\frac{\beta}{2}-\frac{13}{4}}dx_1$$

$$< \frac{C|\log \Delta|^{\gamma}}{(i+1)^{3/2}}(1+x_2)^{\alpha-m+\frac{\beta}{2}-\frac{9}{4}}$$

provided that $\alpha - m + \frac{\beta}{2} - \frac{9}{4} < 0$. Combining (5.19) and (5.25),

$$(5.26)\qquad \left|\frac{d^m b_i(x_2)}{dx_2^m}\right| < C|\log \Delta|^{\gamma}(1+x_2)^{\alpha-m+\frac{\beta}{2}-\frac{5}{4}}\min\left\{(i+1)^{1/2}, \frac{(1+x_2)^{-1}}{(i+1)^{3/2}}\right\}.$$

Using the logarithmic inequality

$$\min\{a,b\} < a^{1/2}b^{1/2}$$

yields (5.20).

Let us analyze now $\rho_p = \psi_p - v_p$ given in (5.17). We have

$$\rho_p(x_1,x_2) = \sum_{i=0}^{p} [b_i(x_2)-b_i^{[p]}(x_2)]P_i(x_1)$$

$$\frac{\partial\rho_p}{\partial x_1} = \sum_{i=1}^{p} [b_i(x_2)-b_i^{[p]}(x_2)]P_i'(x_1).$$

Because

$$P_i'(x,\beta,\beta) = \frac{1}{2}(2\beta+i+1)P_{i-1}(x,\beta+1,\beta+1)$$

(see [12], p. 895 formula 8.961.4), we obtain for $0 \leq m \leq p+1$

$$A_1 = \int_{-1}^{+1}\left(\int_{-1}^{+1}\left(\frac{\partial\rho_p}{\partial x_1}(x_1,x_2)\right)^2(1-x_1^2)^{\beta+1}dx_1\right)(1-x_2^2)^{\beta}dx_2 \tag{5.27}$$

$$\leq C\int_{-1}^{+1}\left(\sum_{i=1}^{p} i(b_i(x_2)-b_i^{[p]}(x_2))^2(1-x_2^2)^{\beta}dx_2\right.$$

$$\leq C\sum_{i=1}^{p} i\int_{-1}^{+1}\left(\sum_{p+1}^{\infty} a_{ij}P_j(x_2)\right)^2(1-x_2^2)^{\beta}dx_2$$

$$= C\sum_{i=1}^{p} i\sum_{j=p+1}^{\infty}\frac{a_{ij}^2}{j} \leq \frac{C}{p^{2m}}\sum_{i=1}^{p} i\sum_{j=p+1}^{\infty}\frac{a_{ij}^2(j+m)!}{(j-m)!j}$$

$$\leq C\frac{1}{p^{2m}}\sum_{i=1}^{p} i\int_{-1}^{+1}\left(\frac{d^m b_i(x_2)}{dx_2^m}\right)^2(1-x_2^2)^{\beta+m}dx_2.$$

Using (5.14) we see that the support of $b_i(x_2)$ lies in $I_2 = [-1+\Delta\sin\theta_0, 0]$ where $\tan\theta_0 = \frac{1}{\kappa_0}$. Hence from (5.27) and (5.20) for $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$

$$(5.28) \qquad A_1 \leq \frac{C}{p^{2m}} \sum_{i=1}^{p} i \int_{-1+\Delta \sin\theta_0}^{0} \left(\frac{d^m b_i}{dx_2^m}\right)^2 (1-x_2^2)^{\beta+m} dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2m}} \sum_{i=1}^{p} i \int_{-1+\Delta \sin\theta_0}^{0} \frac{1}{i} (1+x_2)^{2(\alpha-m+\frac{\beta}{2}-\frac{7}{4})+\beta+m} dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2m-1}} \Delta^{2\alpha-m+2\beta-\frac{5}{2}}$$

provided that $2\alpha - m + 2\beta - \frac{5}{2} < 0$. Analogously, using (5.19) for $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$ and $0 \leq m \leq p + 1$,

$$(5.29) \qquad A_2 = \int_{-1}^{+1} \left(\int_{-1}^{+1} (\rho_p)^2 (1-x_1^2)^{\beta} dx_1\right)(1-x_2^2)^{\beta} dx_2$$

$$\leq C \int_{-1}^{+1} \sum_{i=1}^{p} \frac{1}{i} \left(b_i(x_2) - b_i^{[p]}(x_2)\right)^2 (1-x_2^2)^{\beta} dx_2$$

$$\leq \frac{C}{p^{2m}} \sum_{i=1}^{p} \frac{1}{i} \int_{-1+\Delta \sin\theta_0}^{0} \left(\frac{d^m b_i(x_2)}{dx_2^m}\right)^2 (1-x_2^2)^{\beta+m} dx$$

$$\leq \frac{C}{p^{2m}} \sum_{i=1}^{p} \frac{1}{i} \int_{-1+\Delta \sin\theta_0}^{0} i|\log \Delta|^{2\gamma} (1+x_2)^{2(\alpha-m+\frac{\beta}{2}-\frac{5}{4})} (1+x_2)^{\beta+m} dx_2$$

$$\leq \frac{C}{p^{2m-1}} |\log \Delta|^{2\gamma} \Delta^{2\alpha-m+2\beta-\frac{3}{2}}$$

provided that $2\alpha - m + 2\beta - \frac{3}{2} < 0$.

Similarly we estimate the term σ_p in (5.17) with $0 \leq m \leq p + 1$

$$\sigma_p(x_1,x_2) = \sum_{i=p+1}^{\infty} b_i(x_2) P_i(x_2)$$

$$(5.30) \qquad B_1 = \int_{-1}^{+1} \left(\int_{-1}^{+1} \left(\frac{\partial\sigma_p}{\partial x_1}\right)^2 (1-x_1^2)^{\beta+1} dx_1\right)(1-x_2^2)^{\beta} dx_2$$

$$\leq C \int_{-1}^{+1} \left(\sum_{i=p+1}^{\infty} i b_i^2(x_2)\right)(1-x_2^2)^{\beta} dx_2$$

$$\leq \frac{C}{p^{2(m-1)}} \int_{-1}^{+1} \sum_{i=p+1}^{\infty} \frac{b_i^2(x_2)(i+m)!}{(i-m)!i} (1-x_2^2)^{\beta} dx_2$$

$$\leq \frac{C}{p^{2(m-1)}} \int_{-1}^{+1} \left(\int_{-1}^{+1} \left(\frac{\partial^m v}{\partial x_1^m}\right)^2 (1-x_1^2)^{\beta+m} dx_1\right)(1-x_2^2)^{\beta} dx_2.$$

Since the support of v lies in $R_{\kappa_0} - S_{\kappa_0}^{\Delta}$ we can use Lemma 5.1 and obtain with $I_1 = [-1 + \frac{1}{\kappa_0}(1+x_2), \ -1 + \kappa_0(1+x_2)]$

$$(5.31) \qquad B_1 \leq C \frac{|\log \Delta|^{2\gamma}}{p^{2(m-1)}} \int_{-1+\Delta \sin\theta_0}^{0} \int_{I_1} (1+x_1)^{2(\alpha-2-m)} (1-x_1^2)^{\beta+m} (1-x_2^2)^{\beta} dx_1 dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2(m-1)}} \int_{-1+\Delta \sin \theta_0}^{0} (1+x_2)^{2\alpha-m+2\beta-3} dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2(m-1)}} \Delta^{2\alpha-m+2\beta-2}$$

provided that $2\alpha - m + 2p - 2 < 0$.

Similarly for $0 \leq m \leq p + 1$

$$(5.32) \qquad B_2 = \int_{-1}^{+1} \int_{-1}^{+1} (\sigma_p)^2 (1-x_1^2)^{\beta} (1-x_2^2)^{\beta} dx_1 dx_2$$

$$\leq \frac{C}{p^{2m}} \int_{-1}^{+1} \int_{-1}^{+1} \left(\frac{\partial^m v}{\partial x_1}\right)^2 (1-x_1^2)^{\beta+m} (1-x_2^2)^{\beta} dx_1 dx_2$$

$$\leq \frac{C}{p^{2m}} |\log \Delta|^{2\gamma} \Delta^{2\alpha-m+2\beta-2}$$

provided that $2\alpha - m + 2\beta - 2 < 0$.

We will summarize (5.29) - (5.32).

LEMMA 5.5. Let ρ_p and σ_p be defined in (5.17). Then for $0 \leq m \leq p + 1$ and $k = 0,1$

$$(5.33) \qquad \int_{-1}^{+1} \int_{-1}^{+1} \left(\frac{\partial^k \rho_p}{\partial x_1^k}\right)^2 (1-x_1^2)^{\beta+k} (1-x_2^2)^{\beta} dx_1 dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2m-1}} \Delta^{2\alpha-m+2\beta-\frac{3}{2}-k}$$

provided that $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$ and $2\alpha - m + 2\beta - \frac{3}{2} - k < 0$

$$(5.34) \qquad \int_{-1}^{+1} \int_{-1}^{+1} \left(\frac{\partial^k \sigma_p}{\partial x_1^k}\right)^2 (1-x_1^2)^{\beta+k} (1-x_2^2)^{\beta} dx_1 dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2(m-k)}} \Delta^{2\alpha-m+2\beta-2}$$

provided that $2\alpha - m + 2\beta - 2 < 0$. The constant C is independent of Δ, p but depends on α, β, γ, m.

Let

$$R_\kappa^\Delta = R_\kappa \cap Q_0^{2\Delta}$$

where

$$Q_0^{2\Delta} = \{x_1 > -1 + 2\Delta, \quad x_2 > -1 + 2\Delta\}.$$

Further, for $f(x_1,x_2)$, $x_1,x_2 \in Q$ and $\Delta < \frac{1}{4}$ we define

$$f_\Delta(x_1,x_2) = f(x_1-2\Delta,x_2-2\Delta), \quad (x_1,x_2) \in Q_0^{2\Delta} \tag{5.35}$$

and $f_\Delta(x_1,x_2) = 0$ on $Q - Q_0^{2\Delta}$.

LEMMA 5.6. Let $\xi(x_1,x_2)$ be given by (5.1) and $0 < \Delta < \frac{1}{4}$. Then on $R_{\kappa_0}^{\Delta}$

$$|\xi_\Delta(x_1,x_2)| \leq C(1-x_1^2)(1-x_2^2) \tag{5.36}$$

$$\left|\frac{\partial \xi_\Delta}{\partial x_1}(x_1,x_2)\right| \leq C(1-x_1^2). \tag{5.37}$$

Proof. For $(x_1,x_2) \in R_{\kappa_0}$

$$\frac{1}{\kappa_0} \leq \frac{x_2+1}{x_1+1} \leq \kappa_0$$

and for $(x_1,x_2) \in R_{k_0}^{\Delta}$

$$2\Delta \leq 1 + x_1.$$

Hence

$$|\xi_\Delta(x_1 x_2)| = \tag{4.38}$$

$$= |(1+x_1-2\Delta-\kappa(1+x_2-2\Delta))(\kappa(1+x_1-2\Delta) - (1+x_2-2\Delta))|$$

$$= |(1+x_1) - \kappa(1+x_2) + 2\Delta(\kappa-1)||\kappa(1+x_1) - (1+x_2) + 2\Delta(1-\kappa)|$$

$$\leq (|\kappa_0(1+x_2)| + |\kappa(1+x_2)| + |(\kappa-1)\kappa_0(1+x_2)|)$$

$$\times \left(|\kappa(1+x_1)| + |\kappa_0(1+x_1)| + |(\kappa-1)(1+x_1)|\right)$$

$$\leq (1+x_2)(1+x_1)(\kappa_0+\kappa+(\kappa-1)\kappa_0)(\kappa+\kappa_0+\kappa-1).$$

Because on R_{κ_0}

$$(1+x_2) \leq C(1-x_2^2)$$

$$(1+x_1) \leq C(1-x_1^2)$$

(5.38) yields immediately (5.36). (5.37) can be proven in an analogous way.

LEMMA 5.7. Let v satisfy (5.9) and v_p be given by (5.12). Then for $\Delta = p^{-2}$

(5.39) $$\|\xi_\Delta(v-v_p)\|_{H^1(R^\Delta_{\kappa_0})} \leq C|\log p|^\gamma p^{-2\alpha}$$

where C is independent of p.

<u>Proof</u>. As in (5.17) $v - v_p = \sigma_p + \rho_p$. Let us estimate first $\|\xi_\Delta\rho_p\|_{H^1(R^\Delta_{\kappa_0})}$. To this end we estimate $\|\xi_\Delta\rho_p\|_{L_2(R^\Delta_{\kappa_0})}$, $\|\frac{\partial\xi_\Delta}{\partial x_1}\rho_p\|_{L_2(R^\Delta_{\kappa_0})}$ and $\|\xi_\Delta \frac{\partial\rho_p}{\partial x_1}\|_{L_2(R^\Delta_{\kappa_0})}$. The estimates of the terms involving $\frac{\partial}{\partial x_2}$ follow from the symmetry of x_1 and x_2.

Using (5.36) we get

(5.40) $$D_1 = \|\xi_\Delta \frac{\partial\rho_p}{\partial x_1}\|^2_{L^2(R^\Delta_{\kappa_0})} = \iint_{R^\Delta_{\kappa_0}} \xi^2_\Delta \left(\frac{\partial\rho_p}{\partial x_1}\right)^2 dx_1\,dx_2$$

$$\leq C \iint_{R^\Delta_{\kappa_0}} (1-x_1^2)^2 (1-x_2^2)^2 \left(\frac{\partial\rho_p}{\partial x_1}\right)^2 dx_1\,dx_2.$$

Because on $R^{\Delta}_{\kappa_0}$,

$$0 < C \leq \frac{1-x_1^2}{2\Delta},$$

using Lemma 5.6 we get for $\beta > 2$

$$D_1 \leq C \int_{-1}^{+1}\int_{-1}^{+1} \frac{(1-x_1^2)^{\beta+1}}{\Delta^{\beta-1}} \frac{(1-x_2^2)^{\beta}}{\Delta^{\beta-2}} \left(\frac{\partial \rho_p}{\partial x_1}\right)^2 dx_1 dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2m-1}} \Delta^{2\alpha-m+2\beta-\frac{5}{2}-2\beta+3}$$

$$= C\, |\log \Delta|^{2\gamma} \frac{\Delta^{2\alpha-(m-1/2)}}{p^{2(m-1/2)}}$$

provided that $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$ and $2\alpha - m + 2\beta - \frac{5}{2} < 0$.

Choosing m large enough and $\Delta = p^{-2}$ we get

$$D_1 \leq C|\log p|^{2\gamma} p^{-4\alpha}. \tag{5.41}$$

Similarly

$$D_2 = \left\| \frac{\partial \xi_\Delta}{\partial x_1} \rho_p \right\|^2_{L_2(R^{\Delta}_{\kappa_0})}$$

$$\leq C \iint_{R^{\Delta}_{\kappa_0}} (1-x_1^2)^2 (\rho_p)^2\, dx_1 dx_2$$

$$\leq C \int_{-1}^{+1}\int_{-1}^{+1} \frac{(1-x_1^2)^{\beta}}{\Delta^{\beta-2}} \frac{(1-x_2^2)^{\beta}}{\Delta^{\beta}} (\rho_p)^2\, dx_1 dx_2$$

$$\leq C \frac{|\log \Delta|^{2\gamma}}{p^{2m-1}} \Delta^{2\alpha-m+2\beta-\frac{3}{2}-2\beta+2}$$

$$= C|\log \Delta|^{2\gamma} \frac{\Delta^{2\alpha-m+1/2}}{p^{2(m-1/2)}}$$

provided that $\alpha - m + \frac{\beta}{2} - \frac{5}{4} < 0$ and $2\alpha - m + 2\beta - \frac{3}{2} < 0$. Hence for m large enough

$$D_2 \leq C|\log p|^{2\gamma} p^{-4\alpha}. \tag{5.42}$$

Also

$$D_3 = \|\xi_\Delta \rho\|_{(R^\Delta_{\kappa_0})} \leq \int_{-1}^{+1}\int_{-1}^{+1} \frac{(1-x_1^2)^\beta (1-x_2^2)^\beta}{\Delta^{2\beta}} \rho_p^2 dx_1 dx_2 \tag{5.43}$$

$$\leq C|\log p|^{2\gamma} p^{-4\alpha-4}.$$

Combining (5.41), (5.42), (5.43) we get

$$\|\xi_\Delta \rho_p\|_{H^1(R^\Delta_{\kappa_0})} \leq C|\log p|^\gamma p^{-2\alpha}. \tag{5.44}$$

The estimate for $\|\xi_\Delta \sigma_p\|_{H^1(R^\Delta_{\kappa_0})}$ can be obtained quite analogously.

Let now $\alpha > 1$ not be an integer and $k = [\alpha]$ be the largest integer less than α. For $0 \leq \tau \leq k$, τ integer, let $v^{[\tau]}$ denote the $\tau^{\underline{th}}$ derivative of v (see 5.7a) along the direction $\hat{n}$ where $\hat{n}$ is the unit vector along the line $x_2 = x_1$. Then we see that $v^{[\tau]}$ satisfies Lemma 5.1 with α replaced by $\alpha - \tau > 0$. Hence using Lemma 5.7 we get

$$(5.45) \qquad \|\xi_\Delta(v^{[\tau]}-v_p^{[\tau]})\|_{H^1(R^\Delta_{\kappa_0})} \leq C|\log p|^\gamma p^{-2(\alpha-\tau)}.$$

Let ω^Δ be defined by (5.6) and ω^Δ_Δ be its translation given by (5.35). Then (see 5.8)

$$(5.46) \qquad u_{0\Delta} = u_{0\Delta}\omega^\Delta_\Delta + u_{0\Delta}(1-\omega^\Delta_\Delta)$$

$$= v_\Delta + w_\Delta.$$

Because $u \in H^1_0(R_{\kappa_0})$, then $u_\Delta \in H^1_0(R_{\kappa_0})$ and hence

$$\xi_\Delta w_\Delta = u_\Delta(1-\omega^\Delta_\Delta) \in H^1(R^\Delta_{\kappa_0}).$$

LEMMA 5.8. Let $\Delta = p^{-2}$, $\tilde{\Delta} = 2\sqrt{2}\,\Delta$. Then for $k = [\alpha]$, $\alpha > 0$ noninteger

$$(5.47) \qquad \|\xi_\Delta(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]})\|_{H^1(R^\Delta_{\kappa_0})} \leq C|\log p|^\gamma p^{-2\alpha}$$

$$(5.48) \qquad \|\xi_\Delta w_\Delta\|_{H^1(R^\Delta_{\kappa_0})} \leq C|\log p|^\gamma p^{-2\alpha}$$

where C does not depend on p, Δ.

__Proof__. By Taylor's theorem, for any $(x_1,x_2) \in R^\Delta_{\kappa_0}$ and $s = 0,1$, we get using Lemma 5.1

$$\left|\frac{\partial^s}{\partial x_1^s}\left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v^{[i]}\right)(x_1,x_2)\right|$$

$$\leq C\,\Delta^{k+1}\left|\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^{k+1} \frac{\partial^s v}{\partial x_1}(x_1-\theta,x_2-\theta)\right|$$

$$\leq C\Delta^{k+1}(1+x_1)^{\alpha-3-s-k}|\log \Delta|^{\gamma}$$

where $0 < |\theta| < 2\Delta$. Hence using Lemma 5.6 we get for $\Delta = p^{-2}$

$$\left\| \frac{\partial^{1-s}}{\partial x_1^{1-s}} (\xi_\Delta) \frac{\partial^s}{\partial x_1^s} \left(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}_i}{i!} v^{[i]}\right)\right\|^2_{L^2(R^\Delta_{\kappa_0})}$$

$$\leq C \iint_{R^\Delta_{\kappa_0}} (1-x_1^2)^2 (1-x_2^2)^{2s} \Delta^{2(k+1)} |\log \Delta|^{2\gamma} (1+x_1)^{2\alpha-6-2s-2k}$$

$$\leq C\Delta^{2(k+1)} |\log \Delta|^{2\gamma} \int_{2\Delta}^{1} (1+x_1)^{2(\alpha-k-2)+1} dx_1$$

$$\leq C|\log \Delta|^{2\gamma} \Delta^{2(k+1)+2(\alpha-k-1)}$$

$$\leq C|\log p|^{2\gamma} p^{-4\alpha}.$$

In the above inequality we used the obvious fact that $\alpha - k - 1 < 0$. The other terms in (5.47) can be bounded analogously and the first part of Lemma 5.8 is proven.

Let us now prove (5.48). It is easy to see that

$$\|\xi_\Delta w_\Delta\|_{H^1(R^\Delta_{\kappa_0})} = \|\xi w\|_{H^1(T)}$$

where $T = \{(r,\theta)|,\ \ 0 < r < 2\Delta,\ \ 0 < \theta < \frac{\pi}{2}\}$. Using (5.1) we have

(5.49) $$|\xi(r,\theta)| \leq Cr^2$$

(5.50) $$\left|\frac{\partial \xi}{\partial r}(r,\theta)\right| \leq Cr.$$

Further, by (5.3)

$$(5.51)\quad \begin{aligned} |w(r,\Theta)| &< |u_0(r,\Theta)| < C|\log r|^{\gamma} r^{\alpha-2} && \text{for } r < 2\Delta \\ &= 0 && \text{for } r > 2\Delta \end{aligned}$$

$$(5.52)\quad \begin{aligned} \left|\frac{\partial w(r,\Theta)}{\partial r}\right| &< C|\log r|^{\gamma} r^{\alpha-3} && \text{for } r < 2\Delta \\ &= 0 && \text{for } r > 2\Delta. \end{aligned}$$

Hence, since $\alpha > 0$ for $\Delta = p^{-2}$:

$$\begin{aligned} \left\| \xi \frac{\partial w}{\partial r} \right\|^2_{H^1(T)} &= \int_0^{\pi/2} \int_0^{2\Delta} \xi^2 \left|\frac{\partial w}{\partial r}\right|^2 r \, dr \, d\Theta \\ &< C \int_0^{2\Delta} |\log r|^{2\gamma} r^{2\alpha-6+4+1} dr \, d\Theta \\ &< C|\log \Delta|^{2\gamma} \Delta^{2\alpha} \\ &< C|\log p|^{2\gamma} p^{-4\alpha}. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\| \frac{\partial \xi}{\partial r} w \right\|^2_{L^2(T)} &< C \int_0^{2\Delta} |\log r|^{2\gamma} r^{2\alpha-5} dr \\ &< C|\log p|^{2\gamma} p^{-4\alpha}. \end{aligned}$$

The other terms in (5.48) can be treated in a similar way. Hence Lemma 5.8 is completely proven.

Now we can prove our main result.

Proof of Theorem 5.1

Let $\hat{S}_\kappa$ be the translation of S_κ obtained by the transformation $\hat{x}_i = x_i - 2\Delta$. Let

$$z_{p\Delta} = \xi_\Delta\Big(\sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}^i}{i!} v_p^{[i]}\Big), \qquad k = [\alpha].$$

Then $z_{p\Delta} \in P_{p+2}(Q)$ and $z_{p\Delta} = 0$ on the sides of $\hat{S}_\kappa$. We have

$$\|u_\Delta - z_{p\Delta}\|_{H^1(R^\Delta_{\kappa_0})} = \|\xi_\Delta\Big(u_{0\Delta} - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}_i}{i!} v_p^{[i]}\Big)\|_{H^1(R^\Delta_{\kappa_0})}$$

$$\leq \|\xi_\Delta w_\Delta\|_{H^1(R^\Delta_{\kappa_0})} + \|\xi_\Delta\Big(v_\Delta - \sum_{i=0}^{k} (-1)^i \frac{\tilde{\Delta}_i}{i!} v^{[i]}\Big)\|_{H^1(R^\Delta_{\kappa_0})} +$$

$$+ \sum_{i=0}^{k} \frac{\tilde{\Delta}_i}{i!} \|\xi_\Delta\big(v^{[i]} - v_p^{[i]}\big)\|_{H^1(R^\Delta_{\kappa_0})}$$

by (5.45), (5.47) and (5.48)

$$\leq C|\log p|^\gamma p^{-2\alpha}.$$

Translating back to S_κ we get the theorem.

Remark 5.1. We have proven more than Theorem 5.1 claims. It is sufficient to assume that v and w defined by (5.7a), (5.7b) satisfy (5.9) and (5.51), (5.52) respectively.

Remark 5.2. It is easy to see from the proof that the internal angle ω_i of γ_i and γ_{i+1} could be equal to 2π, i.e. that we may also consider the slit domain.

5.2 The rate of convergence

We return now to the problem of approximation of the function $u_3^{[i]}$ in (2.4).

Let the vertex A_i be at the origin O. The part of the domain Ω containing the elements with vertices at O is shown in Fig. 5.2.

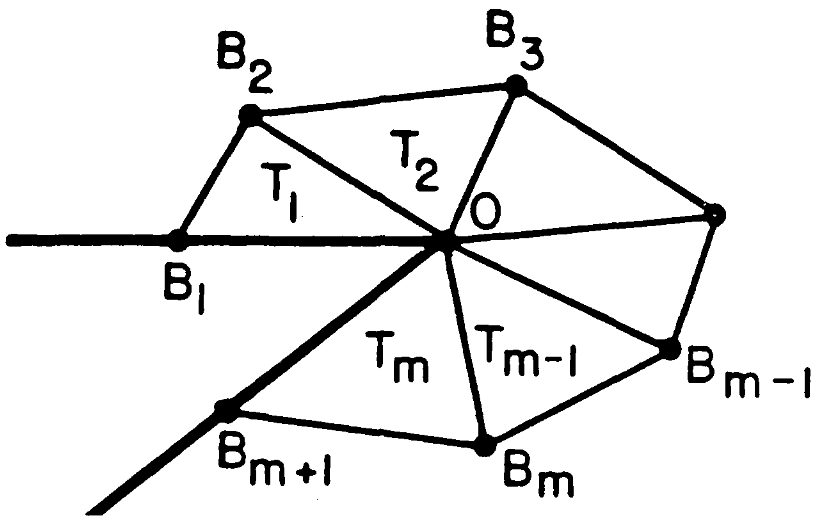

Fig. 5.2

We will assume that the lines $\overline{OB_j}$ have the coordinate Θ_j and that $\overline{OB_1} \subset \Gamma^D$, $\overline{OB_{m+1}} \subset \Gamma^N$. Although we assume that we have only triangular elements, the case when elements are parallelograms does not change the argument.

Let $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{T}_i$, $\bigcup_{i=2}^{m} \overline{B_iB_{i+1}} = \tilde{\Gamma}$. Denote $\mathcal{D}_\rho = \{x \mid x_1^2 + x_2^2 < \rho\}$ and assume that $\mathcal{D}_{\rho_0} \subset \Omega$, $0 < \rho_0 \leq 1$. Now we prove

THEOREM 5.2. Let u be the function given by (5.2) with $\rho \leq \rho_0\mu$ and μ sufficiently small. Then there exists $z_p \in H^1(\Omega)$ satisfying $z_p \in P_p(T_i)$, $i = 1,\ldots,m$, $z_p = 0$ on OB_1 and $\tilde{\Gamma}$ and

(5.53) $$\|u-z_p\|_{H^1(\Omega)} \leq C|\log p|^\gamma p^{-2\alpha}$$

where C is independent of p.

Proof.

1. Assume first that $\Phi(\theta_j) = 0$, $j = 1,\ldots,m+1$. Denote $\Phi_j(\theta)$ to be an extension of $\Phi(\theta)$ onto $(\theta_j-\Delta,\theta_{j+1}+\Delta)$ where $|2\Delta+\theta_{j+1}-\theta_j| < \pi$ and $\Phi_j(\theta) = 0$ on $(\theta_j-\Delta,\theta_j-\frac{\Delta}{2})$ and $(\theta_j+\frac{\Delta}{2},\theta_j+\Delta)$

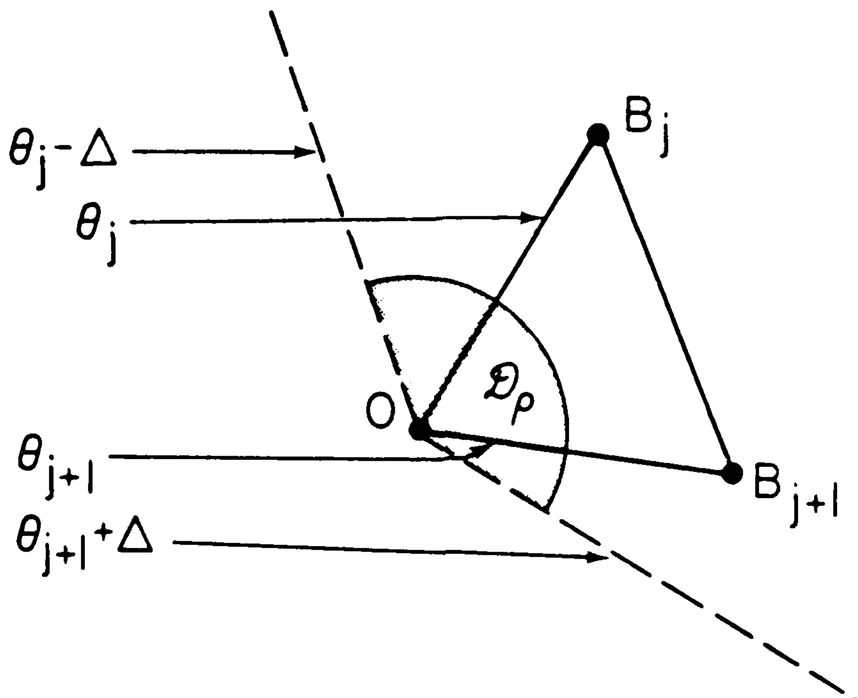

Fig. 5.3

Let $\tilde{S} = \{(r,\theta)|\theta_j-\Delta < \theta < \theta_{j+1}+\Delta\}$ and $\tilde{S}^* = \{(r,\theta)|\theta_j < \theta < \theta_{j+1}\}$. Denote by u_j the function given by (5.2) when $\Phi(\theta)$ is replaced by $\Phi_j(\theta)$ and extend u_j by zero. Let now T be the linear mapping which maps $\tilde{S}$ onto R_{κ_0} and $\tilde{S}^*$ onto R_κ. Denote $\tilde{T}_j = T(T_j)$ and assume that $\tilde{T}_j \subset \tilde{R}_\kappa \subset \tilde{Q}_0$.

The mapping T transforms u_j into $\tilde{u}_j$ on $\tilde{T}_j$. Denoting by η_j the linear function which is zero on $T(\overline{B_jB_{j+1}})$, the function $\frac{\tilde{u}_j}{\eta_j}$ satisfies obviously the conditions mentioned in the Remark 5.1 to Theorem 5.1. Therefore, $\frac{\tilde{u}_j}{\eta_j}$ can be approximated by a function z_p^* satisfying (5.53) on

$\tilde{T}_j$ and hence $z_p^* \eta_j = z_{p+1}^*$ satisfies (5.53) too. Hence (5.53) is proven in the case that $\Phi(\Theta_j) = 0$, $j = 1,\ldots,m$.

2. Now we will consider the case when $\Phi(\Theta_j) = 0$ for $j \neq j_0$. Consider the elements T_{j_0-1}, T_{j_0} as shown in Fig. 5.4

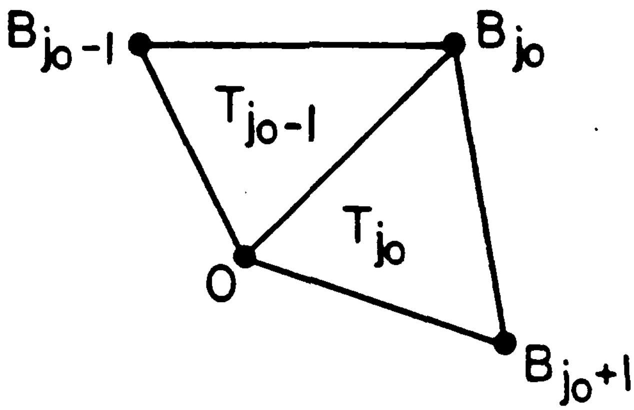

Fig. 5.4

If the angle $|\Theta_{j_0+1} - \Theta_{j_0-1}| < \pi$, then we can proceed exactly in the same way as before only replacing η_j by $\eta_{j_0-1}\eta_{j_0}$. Hence we have to consider the case when $\Theta_{j_0+1} - \Theta_{j_{0-1}} > \pi$ (see Fig. 5.5)

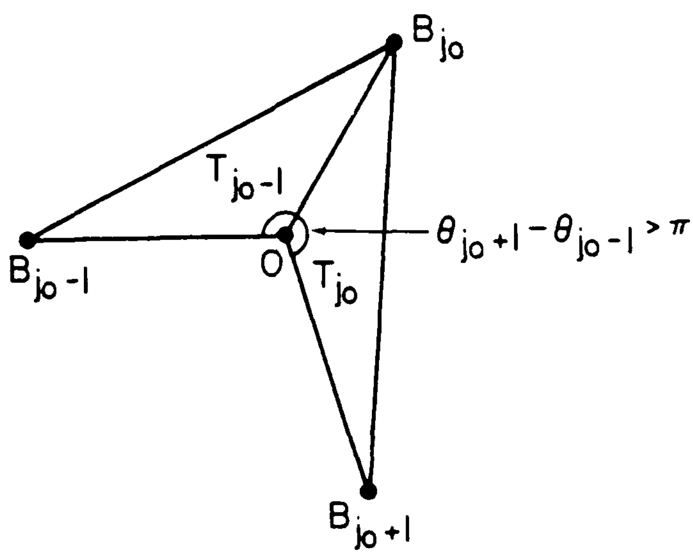

Fig. 5.5

In this case we first map T_{j_0} onto $\tilde{T}_{j_0}$ by a linear mapping so that $\overline{OB}_{j_0}$ is mapped onto itself and the total angle is $< \pi$ (see Fig. 5.6)

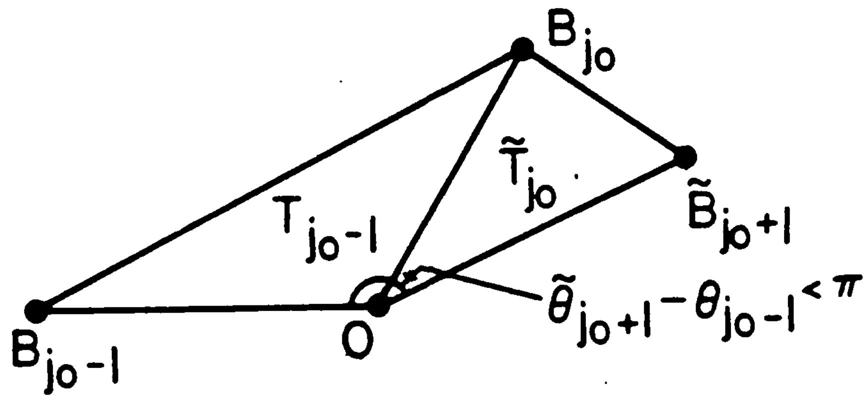

Fig. 5.6

Extending $\Phi(\Theta)$, $\Theta_{j_0-1} < \Theta < \Theta_{j_0}$ so that $\Phi(\tilde{\Theta}_{j_0+1}) = 0$ we can get the desired estimate for this case as before. We approximated on $\tilde{T}_{j_0}$, of course, a different function than we wanted. Nevertheless, the difference is zero at Θ_{j_0} and $\tilde{\Theta}_{j_0+1}$, and we can proceed analogously as in part 1 of the proof. Hence (5.53) is proven.

6. THE CONVERGENCE OF THE p-VERSION OF THE FINITE ELEMENT METHOD

In this section we will summarize the results we have proven and further generalize them.

6.1. The case with triangular and parallelogram elements

We have

THEOREM 6.1. Let Ω be the polygonal domain as introduced in Section 2. Consider the problem (2.2) (2.3) and assume that the solution can be written in the form (2.4) (2.5) with $k > 3/2$ and that Ω is Lipschitz for $q \leq 3/2$.

Assume that u_p is the finite element solution as described in Section 2.3 with triangular or parallelogram elements. Then

$$\|u-u_p\|_{H^1(\Omega)} \leq Cp^{-\mu}|\log p|^{\nu}R \tag{6.1}$$

where

$$\mu = \min_i\left(q-1,\ k-1,\ 2\alpha_1^{[i]}\right) = \min\left(q-1, k-1, 2\alpha_1^{[j]}\right) \tag{6.2a}$$

$$\nu = \begin{cases} \max \gamma_1^{[j]} & \text{if } \mu = 2\alpha_1^{[j]} \\ 0 & \text{otherwise} \end{cases} \tag{6.2b}$$

$$R = \|u_1\|_{H^q(\Omega)} + \|u_2\|_{H^k(\Omega)} + \sum_{i,\ell} |c_\ell^{[i]}|. \tag{6.2c}$$

The theorem follows immediately from Theorem 4.3 and 5.2.

Although we have dealt with the model problem (2.2), (2.3) only, it is obvious that the theorem holds for any second order elliptic problem if the

solution has the form (2.4), (2.5) or when (2.5) is different but has the same character concerning the growth of its derivatives.

6.2. The case of curved elements

So far we assumed that the elements are triangular or parallelogram. The obtained results can be immediately generalized to the case of curvilinear triangles and quadrilaterals which can be mapped individually on the standard triangle or square by a mapping which is one-to-one and sufficiently smooth (in both directions). (In practice this is always achieved.)

We proceed then in the usual manner by approximating the function on standard squares and triangles.

Theorem 6.1 holds then without changes. The function u_3 is of course defined now on the straight line triangles (squares) and does not have the explicit form given in (2.5) but possesses the same character.

So far we assumed that all elements are of the same degree. The modifications in our proofs and results when the degrees are different in different elements are obvious.

REFERENCES

[1] Babuška, I., Dorr, M. R.: Error estimates for the combined h and p version of finite element method. Numer. Math. 37 (1981), 252-277.

[2] Babuška, I., Gui, W., Guo, B., Szabo, B.: Theory and Performance of the h-p Versions of the Finite Element Method. To appear.

[3] Babuška, I., Kellogg, R. B., Pitkaranta, J.: Direct and inverse error estimates for finite elements with mesh refinement. Numer. Math. 33 (1979), 447-471.

[4] Babuška, I., Szabo, B. A., Katz, I. N.: The p-version of finite element method. SIAM J. Numer. Anal., 18 (1981), 515-545.

[5] Babuška, I., Szabo, B. A.: Lecture notes on finite element analysis. In preparation.

[6] Bergh, I., Lofstrom, J.: Interpolation spaces. Berlin-Heidelberg-New York: Springer-Verlag, 1976.

[7] Dorr, M. R.: The approximation theory for the p-version of the finite element method. SIAM J. Numer. Anal. 21 (1984), 1180-1207.

[8] Dorr, M. R.: The approximation theory for the p-version of the finite element method II. SIAM J. Numer. Anal. To appear.

[9] Gui, W., Babuška, I.: The h, p and h-p versions of the Finite Element Method for One Dimensional Problem, Part 1: The Error Analysis of the p-version, Tech. Note BN-1036; Part 2: The Error Analysis of the h and h-p Versions, Tech, Note BN-1037; Part 3: The Adaptive h-p Version. Tech. Note BN-1038, IPST, University of Maryland, College Park, 1985.

[10] Guo, B., Babuška, I.: The h-p version of the Finite Element Method I. To appear.

[11] Guo, B., Babuška, I.: The h-p version of the Finite Element Method II. To appear.

[12] Gradshteyn, I. S., Ryzhik, I. M.: Table of integrals series and products. London & New York: Academic Press, 1965.

[13] Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Boston-London-Melbourne: Pitman, 1985.

[14] Kondrat´ev, V. A.: Boundary-value problems for elliptic equations in domains with conic or cornerpoints, Trudy Moskov. Mat. Obschch. 16 (1967), 209-292. MR 37 #1777. = Trans. Moscow Math. Soc., 1967, 227-313.

[15] Kondrat´ev, V. A. and Oleinik, O. A.: Boundary-value problems for partial differential equations in non-smooth domains, Russian Math. Surveys 38:2 (1983), 1-86.

[16] Suetin, P. K.: Classical Orthogonal Polynomials, Moscow, 1979 (in Russian).

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